Approximating Solution Of Hybrid Differential Equation With Maxima

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Abstract: In this work, we look into hybrid differential equations with maxima and discuss about approximation of the solutions. The primary solution is based on the Dhage iteration technique, which is contained in Dhage's recent hybrid fixed point theorem in a partly ordered normed linear space.

Keywords: Hybrid differential equation (HDE),Differential equation with maxima, Dhage Iteration method, Approximation of solutions.

1. INTRODUCTION

The study of fixed point theorems for the contraction mapping in partially ordered metric spaces is initiated in [21] which is further continued by[19] and applied to periodic boundary value problems of nonlinear differential equations for proving the existence results under certain monotonic conditions, similarly, the study of hybrid fixed point theorems in a partially ordered metric space is initiated by [8,9,11] with applications to nonlinear differential and integral equation under weaker mixed conditions of nonlinearities [10,12] and the references therein. The Dhage iteration technique encapsulated in a hybrid fixed point theorem is used in this work to examine the existence of approximate solutions to certain hybrid differential equations with maxima. The findings of this study are novel in the field of nonlinear differential equations with maxima, according to the authors.

Given a closed and bounded interval $J = [0, T]$of the real line R for some T > 0, we consider the following HDE of order periodic boundary value problems,

$$\left.\begin{aligned} x'(t) = f\left(t, \max_{0\le\xi\le t} x(\xi)\right) + g\left(t, \max_{0\le\xi\le t} x(\xi)\right), \quad t\in J \\ x(0) = x(T) \end{aligned}\right\} \qquad (1.1)$$

For all $t \in J = [0,T]$and $f, g : J \times R \to R$ are continuous functions.

By solutions of equation (1.1) we mean a function $x \in C^1(J, R)$ that satisfies equation (1.1), where $C^1(J,R)$ is the space of continuously differential real- valued functions defined on J .Differential equations with maxima are useful in the theory of automated control System applications. There have been several results on existence and uniqueness, asymptotic stability, and numerical solutions for such equations [1, 20].

The Periodic Boundary Value Problem's ofnonlinear first order ordinary differential equations have also been a topic of a great interest since long time. The Hybrid Differential Equation (1.1) is a linear perturbation of first type of the Periodic Boundary Value Problem of First order nonlinear differential equations. The details of different types of perturbation appear in [13]. The Special cases of Hybrid Differential Equation (1.1) are

$$\left.\begin{aligned} x'(t) &= f\left(t, \max_{0 \le \xi \le t} x(\xi)\right), \quad t \in J \\ x(0) &= x(T) \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} x'(t) &= g\left(t, \max_{0 \le \xi \le t} x(\xi)\right), \quad t \in J \\ x(0) &= x(T) \end{aligned}\right\} \tag{1.3}$$

The hybrid differential equation (1.3) has been studied earlier using Picards method [1].Very recentlyauthors [20]have initiated the study of initial value problems of first order ordinary nonlinear differential equations via new Dhage iteration method. However, to the best of author's knowledge the Hybrid Differential Equation (1.3) is not discussedvia Dhage Iteration method. Therefore, Hybrid Differential Equation (1.1) is new to the iteration in the set up of Dhage iteration method. In this paper we discuss the Hybrid Differential Equation (1.1) for approximation of the solutions via a new approach based upon Dhage iteration method which include the existence and approximation results for Hybrid Differential Equations (1.2) and (1.3) as special cases which are again new to the theory of differential equations.The next part contains some preliminaries as well as the primary tool that will be used to prove the paper's major finding.

2. PRELIMINARIES

Throughout this paper,let$(E, \preccurlyeq, \|.\|)$denote a partially ordered normed linear space. Two elements x and $y$in E are said to be comparable if either the relation $x \preccurlyeq y$ or $y \succcurlyeq x$ holds.Elements of C are comparable if C is totally ordered (that is chain)

It is known that E is regular if $\{x_n\}$ is a nondecreasing (resp.nonincreasing) sequence in E such that $x_n \to x^*$ as n$\to \infty$, then $x_n \preccurlyeq x^*$(resp.$x_n \succcurlyeq x^*$) for all $n \epsilon N$.The conditions guaranteeing the regularity of E may be found in[18].

We need the following definitions[8,9]as follows.

Definition 2.1 [9]:A mapping $\tau : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\preccurlyeq$, that is , if x $\preccurlyeq y$ implies τx$\preccurlyeq$ τy for all $x, y \in E$. Similarly, τ is called monotone nonincreasing if x$\preccurlyeq y$ implies $\tau x \succcurlyeq \tau y$ for all $x, y \in E$. If τ is called monotonic if it is either monotone nondecreasing or monotone nonincreasing on E.

Definition 2.2[9]:A mapping $\tau: E \to E$ is called partially continuous at a point $a \epsilon E$ if for $\in > 0$ there exists a $\delta > 0$ such that$\| \tau x - \tau y \| < \in$ whenever is comparable to a and$\|x - a\| < \delta$. τ is called partially continuous on E if it is partially continuous at every point of it.It is clear that if τ is partially continuous on E,then it is partially continuous on every chain Ccontained in E.

Definition2.3 [9]:A nonempty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. An operator τ on partially normed linear space E into itself is called partially bounded if $\tau(E)$ is a partially bounded subset of E.τ is called uniformly partially bounded if all chains C in $\tau(E)$ are bounded by a unique constant.

Definition 2.4 [9]: A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is relatively compact subset of E.A mapping τ:$E \to E$ is called partially compact if $\tau(E)$ is a partially relatively compact subset of E. τ is called uniformly partially compact if τ is uniformly partially bounded and partially compact operator on E.τ is called partially totally bounded subset S of E, $\tau(S)$ is a partially relatively compact subset of E. If τ is partially continuous and partially totally bounded,then it is called completely continuous on E.

Remark 2.1: Suppose that τ is a nondecreasing operator on E into itself. Then τ is partially bounded or partially compact if $\tau(C)$ is bounded or relatively compact subset of E for each chain C in E.

Definition 2.5 [9]: The order relation $\preccurlyeq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}$ is a monotone sequence, that is,monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ convergesto x^* implies that the original sequence $\{x_n\}$ converges to x^*.Similarly, given a partially ordered normed linear space $(E, \preccurlyeq, \|.\|)$, the order relation $\preccurlyeq$ and the norm $\|.\|$ are said to compatible if $\preccurlyeq$ and the metric d defined through the norm $\| \,.\, \|$ are compatible. A subset S of E is called Janhavi if the order relation $\preccurlyeq$ and the metric d or norm $\|.\|$ are compatible in it.If $S = E$, then E is referred to as Janhavi Banach space.
Particularly, the set R of real numbers with usual order relation $\leq$ and the norm given by the absolute value function $|\,.\,|$ has this characteristic. Similarly, the finite dimensional Euclidean space R^n with usual component wise order relation and the standard norm possesses the compatibility property and so is a Janhavi Banach space.

Definition 2.6 [9]: An upper semi-continuous and monotone nondecreasing function $\psi : R_+ \to R_+$ is called a $D-$ function provided $\psi(0) = 0$. An operator $T : E \to E$ is called partially nonlinear $D-$ contraction if there exists a D - function ψ such that

$$\|Tx - Ty\| \leq \psi(\|x - y\|) \tag{2.1}$$

For all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, $\psi(r) = kr$, $k > 0$, T is called a partial linear contraction on E with a contraction constant k.

The Dhage iteration method embodied in the following applicable hybrid fixed point theorem of [9] in a partially ordered normed linear space is used as a key tool for our work contained in [10,12,15].

Theorem 2.1 [9]: Let $(E, \preccurlyeq, \|\,.\,\|)$ be a regular partially ordered complete normed linear space such that every compact chain C of E is Janhavi. Let $A, B : E \to E$ be two nondecreasing operators such that

(a) A is partially bounded and partially nonlinear $D-$ contraction

(b) B is partially continuous and partially compact, and

(c) There exists an element $x_0 \in E$ such that $x_0 \prec Ax_0 + Bx_0$ or $x \succ Ax_0 + Bx_0$.

Then the operator equation $Ax + Bx = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iteration defined by $x_{n+1} = Ax_n + Bx_n$, $n = 0,1,2,3,.......$ converges monotonically

3. MAIN RESULT

Here, we prove approximation result for Hybrid DifferentialEquation (1.1) on a closed and bounded interval $J = [0\,, T]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the Hybrid Differential Equation (1.1) in the function space $C(J, R)$ of continuous real-valued functions defined on J. We define norm $\|\,.\,\|$ and the order relation $\leq$ in $C(J, R)$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

And

$$x \le y \Leftrightarrow x(t) \le y(t) \quad \text{for all } t \in J \tag{3.2}$$

Clearly, $C(J,R)$ is a Banach space with respect to above norm defined in (3.1) and also partially ordered w.r.to the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J,R)$ is regular amd lattice so that every pair of elements of E has a lower and an upper bound in it. The following useful lemma concering the Janhavi Subsets of $C(J,R)$ follows immediately form the Arzela-Ascoli theorem for compactness.

Lemma 3.1: *Let* $(C(J,R), \le, \|.\|)$ be a partially ordered Banach space with the norm $\|\,.\,\|$ and the orderd relation $\le$ defined by (3.1) & (3.2) respectively. Then every partially compact subset of $C(J,R)$ is Janhavi.

Proof: The proof of lemma is well known and appears in [10,12,14,15].

Lemma 3.2: For any function $\sigma \in L^1(J,R)$, x is a solution to the differential equation

$$\left.\begin{aligned} \frac{dx}{dt} + \lambda x(t) = \sigma(t),\ \ t \in J \\ x(0) = x(t) \end{aligned}\right\} \tag{3.3}$$

It is a solution of the integral equation if and only if

$$x(t) = \int_0^T G_\lambda(t,\ s)\sigma(s)\, ds \tag{3.4}$$

Where

$$G_\lambda(t,s) = \begin{cases} \dfrac{e^{\lambda s - \lambda t + \lambda T}}{e^{\lambda T} - 1}, & if\ 0 \le s \le t \le T, \\ \dfrac{e^{\lambda s - \lambda t}}{e^{\lambda T} - 1}, & 0 \le t \le s \le T \end{cases} \tag{3.5}$$

Notice that the Green's function G_λ is continuous and nonnegative on $J \times J$ and therefore, the number $K_\lambda := \max\ \{|G_\lambda(t,s)| : t,s \in [0,\ T]\}$ exists for all $\lambda \in R^+$.

For the sake of convenience, we write $G_\lambda(t,s) = G(t,s)$ *and* $K_\lambda = K$.

Lemma 3.3: If there exists a differentiable function $u \in C(J,R)$ such that

$$\left.\begin{aligned} u'(t) + \lambda u(t) \le \sigma(t),\ \ t \in J \\ u(0) \le u(T) \end{aligned}\right\} \tag{3.6}$$

For all

$$t \in J,\ Where\ \lambda \in R,\ \lambda > 0\ \ and\ \ \sigma \in L^1(J,R),\ then$$

$$u(t) \le \int_0^T G(t,s)\sigma(s)ds, \tag{3.7}$$

For all $t \in J$, where $G(t,s)$ is a Green's function given by (3.5)

Proof: Suppose that the function $u \in C(J,R)$ satisfies the inequality given in (3.6).

Multiplying the first inequality in (3.6) by $e^{\lambda t}$,

$$\left(e^{\lambda t} u(t)\right)' \le e^{\lambda t} \sigma(t)$$

A direct integration of above inequality from 0 *to* t yields

$$e^{\lambda t}\, u(t) \le u(0) + \int_0^t e^{\lambda s}\, \sigma(s) ds \quad \text{for all } t \in J. \tag{3.8}$$

Therefore, in particular

$$e^{\lambda t}\, u(t) \le u(0) + \int_0^T e^{\lambda s}\, \sigma(s) ds \tag{3.9}$$

Now $u(0) \le u(T)$, *so one has*

$$u(0)\, e^{\lambda T} \le u(T)\, e^{\lambda T} \tag{3.10}$$

From (3.9) and (3.10) it follows that

$$e^{\lambda t}\, u(0) \le u(0) + \int_0^T e^{\lambda s}\, \sigma(s) ds \tag{3.11}$$

Which further yields

$$u(0) \le \int_0^T \frac{e^{\lambda s}}{\left(e^{\lambda T} - 1\right)} \sigma(s)\, ds \tag{3.12}$$

Substituting (3.12) in (3.8) we obtain

$u(t) \le \int_0^T G(t,s)\sigma(s) ds$ for all $t \in J$.

Hence the proof is completed.
For getting result we required the following definition.

Definition 3.1: A function $u \in C^1(J, R)$ is said to be a lower solution of the equation (1.1)
If it satisfies

$$\begin{cases} u'(t) \le f(t, u(t)) + g\left(t, \max\limits_{o \le \xi \le t} u(\xi)\right) \\ u(0) \le u(T) \end{cases} \tag{3.13}$$

For all $t \in J$. Similarly, differentiable function $v \in C^1(J, R)$ is called an upper solution of Hybrid Differential Equation (1.1)if the above inequality is satisfied with reverse sign.
In what follows, we explore the following set of assumptions:

(A_1) There exists constant $\lambda > 0$, $\mu > 0$ *with* $\lambda \ge \mu$ such that

$$0 \le [f(t,x) + \lambda x] - [f(t,y) + \lambda y] \le \mu(x - y)$$

For all $t \in J$ *and* $x, y \in R$, $x \ge y$. *Moreover* , $\lambda K T < 1$

(A_2) There exists a constant $M_g > 0$ such that $|g(t,x)| \le M_g$ for all $t \in J,\ x \in R$

(A_3) $g(t, x)$ is nondecreasing in x for each $t \in J$.

(A_4) Hybrid Differential Equation (1.1) has a lower solution $u \in C^1(J, R)$

Now we consider the following Hybrid Differential Equation

$$\begin{cases} x'(t)+\lambda x(t)= f\left(t,\max_{0\le \xi\le t} u(\xi)\right)+ g\left(t,\max_{0\le \xi\le t} u(\xi)\right) \\ x(0)=x(T) \end{cases} \tag{3.14}$$

For all $t\in J=[0,T]$, where $f\left(t,\max_{0\le \xi\le t} u(\xi)\right)= f\left(t,\max_{0\le \xi\le t} u\right)+\lambda u, \quad \lambda>0$

Remark3.1: A function $u\in C^1(J,R)$ is a solution of the Hybrid Differential Equation (3.13) if and only if it is solution of the Hybrid Differential Equation (1.1) defined on J .
In the next section, we also explore the following Hypothesis.

(A_5) There exists a constant $M_f>0$ such that $f\left(t,\max_{0\le \xi\le t} u(\xi)\right)\le M_f$ for all $t\in J$ *and* $x\in R$

Lemma 3.4: Suppose that hypotheses $(A_2),(A_3)$ *and* (A_5) hold. Then a function $x\in C(J,R)$ is a solution of Hybrid Differential Equation (3.3) if and only if it is solution of the nonlinear hybrid integral equation (in short Hybrid Integral Equation)

$$x(t)=\int_0^T G(t,s)\, f\left(s, \max_{0\le \xi\le s} x(\xi)\right) ds+\int_0^T G(t,s) g\left(s, \max_{0\le \xi\le s} x(\xi)\right) ds \quad \text{for all } t\in J$$

(3.15)

Theorem3.1: Suppose that Hypotheses $(A_1)-(A_5)$ hold. Then the Hybrid Differential Equation (1.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximation defined by $x_0=u$

$$x_{n+1}(t)=\int_0^T G(t,s)\, f\left(s, \max_{0\le \xi\le s} x(\xi)\right) ds+\int_0^T G(t,s) g\left(s, \max_{0\le \xi\le s} x(\xi)\right) ds$$

(3.16)

For all $t\in J$ converges monotonically to x^*.

Proof: Set $E=C(J,R)$. Then, in view of Lemma (3.1), every compact chain C in E possesses the compatibility property with respect to the norm $\|\ .\ \|$ and the order relation $\le$ so that every compact chain C in Janhavi in E.

Define two operators A *and* B *on* E *by*

$$Ax(t)=\int_0^T G(t,s)\, f\left(s, \max_{0\le \xi\le s} x(\xi)\right) ds, \quad t\in J \tag{3.17}$$

and

$$Bx(t)=\int_0^T G(t,s) g\left(s, \max_{0\le \xi\le s} x(\xi)\right) ds, \quad t\in J \tag{3.18}$$

From the continuity of the integral, it follows that A *and* B define the operators $A,B:E\to E$.
Applying Lemma (3.4), the Hybrid Differential Equation (1.1) is equivalent to the operator equation

$$Ax(t)+Bx(t)=x(t), \quad t\in J$$

Now, we show that the operators A *and* B satisfy all the condition of theorem (2.1) in a series of following steps

Step I: A *and* B ***are nondecreasing on*** E **.**

Let $x,y\in E$ be such that $x\ge y$. Then by hypothesis (A_1), *we get*

$$Ax(t) = \int_0^T G(t,s) f\left(s, \max_{0 \le \xi \le s} x(\xi)\right) ds$$
$$\ge \int_0^T G(t,s) f\left(s, \max_{0 \le \xi \le s} x(\xi)\right) ds$$
$$= Ay(t)$$

For all $t \in J$.

Next, we show that the operator B is also nondecreasing on E. Let $x, y \in E$ be such that $x \ge y$. Then $x(t) \ge y(t)$ for all $t \in J$. Since y is continuous on $[0,t]$, there exists a $\xi^* \in [a,t]$ such that $y(\xi^*) = \max_{0 \le \xi \le t} y(\xi)$. By definition of $\le$, one has $x(\xi^*) \ge y(\xi^*)$.

Consequently, we obtain

$$\max_{0 \le \xi \le t} x(\xi) \ge x(\xi^*) \ge y(\xi^*) = \max_{0 \le \xi \le t} y(\xi)$$

Now, using hypothesis (A_3), it can shown that the operator B is also nondecreasing on E.

Step II: A ***is partially contraction and partially bounded on*** E **.**

Let $x \in E$ be arbitrary. Then by (A_5) we have

$$|Ax(t)| \le \int_0^T G(t,s) \left| f\left(s, \max_{0 \le \xi \le s} x(\xi)\right) \right| ds$$
$$\le M_{\tilde{f}} \int_0^T G(t, s) ds$$
$$\le M_{\tilde{f}} KT$$

For all $t \in J$. Taking the supremum over t . We obtain $\|Ax(t)\| \le M_{\tilde{f}} KT$ and so, A is a bounded operator on E. this implies that A is partially bounded on E .

Let $x, y \in E$ such that $x \ge y$ .Then by (A_1) *we have*

$$|Ax(t) - Ay(t)| \le \left| \int_0^T G(t,s) \left[f\left(s, \max_{0 \le \xi \le s} x(s)\right) - f\left(s, \max_{0 \le \xi \le s} y(s)\right) \right] ds \right|$$
$$\le \int_0^T G(t,s) \mu |x(s) - y(s)| ds$$
$$\le \int_0^T G(t,s) \lambda |x(s) - y(s)| ds \text{ ds}$$
$$\le \int_0^T G(t,s) \|x - y\| ds$$
$$\le \lambda KT \|x - y\|$$

for all $t \in J$. Taking the supremum over t, we obtain $\|Ax - Ay\| \le L \| x - y \|$ for all $x, y \in E$ with $x \ge y$, Where $L = \lambda KT < 1$. Hence A is a partially contraction on E and which also implies that A is partially continuous on E .

Step –III: B ***is partially continuous on*** E

Let $\{x_n\}_{n \in N}$ be a sequence in a chain C such that $x_n \to x$, for all $n \in N$. Then

$$\lim_{n \to \infty} Bx_n(t) = \lim_{n \to \infty} \int_0^T G(t,s) g\left(s, \max_{0 \le \xi \le s} x_n(\xi)\right) ds$$

$$= \int_0^T G(t,s)\left[\lim_{n\to\infty} g\left(s, \max_{0 \le \xi \le s} x_n(\xi)\right)\right] ds$$

$$= \int_0^T G(t,s) g\left(s, \max_{0 \le \xi \le s} x_n(\xi)\right) ds$$

$$= Bx(t)$$

For all $t \in J$. this shows that Bx_n converges monotonically to Bx point wise on J.

Now we show that $\{Bx_n\}_{n\in N}$ is an equicontinuous sequence of function in E.

Let $t_1, t_2 \in J \quad with \quad t_1 < t_2$. We have

$$|Bx_n(t_2) - Bx_n(t_1)| = \left| \int_0^T G(t_2,s) g\left(s, \max_{0 \le \xi \le s} x_n(\xi)\right) ds - \int_0^T G(t_1,s) g\left(s, \max_{0 \le \xi \le s} x_n(\xi)\right) ds \right|$$

$$\le \int_0^T |G(t_2,s) - G(t_1,s)| \left| g\left(s, \max_{0 \le \xi \le 0} x_n(\xi)\right) ds \right|$$

$$\le M_g \int_0^T |G(t_2,s) - G(t_1,s)|\, ds \to 0 \quad as \quad t_2 \to t_1$$

Uniformly for all $n \in N$. This shows that the convergence $Bx_n \to Bx$ is uniform and hence B is partially continuous on.

Step IV: The operator B is partially compact on E

Let C be an arbitrary chain in E. We Show that $B(C)$ in uniformly bounded and equicontinous set in E. First we show that $B(C)$ is uniformly bounded.

Let $y \in B(C)$ be any element and Then there is an another element $x \in C$ such that $y = Bx$.

By hypothesis (A_2)

$$|\ y(t)\ | = |Bx(t)|$$

$$= \left| \int_0^T G(t, s) g\left(s, \max_{0 \le \xi \le s} x(\xi)\right) ds \right|$$

$$\le \int_0^T G(t,s) \left| g\left(s, \max_{0 \le \xi \le s} x(\xi)\right) \right| ds$$

$$\le KTM_g = r$$

For $t \in J$. Here we acquire supremum over t we obtain $\|y\| \le \|Bx\| \le r$, for all $y \in B(C)$. Hence $B(C)$ in uniformly bounded subset of E. Next we show that $B(C)$ is an equicontinous set in E

Let $t_1, t_2 \in J, with\ t_1 < t_2$. Then, for any $y \in B(C)$, one has

$$|y(t_2) - y(t_1)| = |\ Bx(t_2) - Bx(t_1)\ |$$

$$= \left| \int_0^T G(t_2,s) g\left(s, \max_{0\le\xi\le S} x(\xi)\right) ds - \int_0^T G(t_1, s)\, g\left(s, \max_{0\le\xi\le s} x(\xi)\right) ds \right|$$

$$\le \int_0^T |G(t_2, s) - G(t_1,s)| \left| g\left(s, \max_{0\le\xi\le s} x(\xi)\right) \right| ds$$

$$\le M_g \int_0^T |G(t_2,s) - G(t_{1,},s)| ds$$

$$\to 0 \quad as \quad t_1 \to t_2$$

Uniformly for all $y \in B(C)$. This shows that $B(C)$ is an equicontinuous subset of E.
So $B(C)$ is a uniformly bounded and equicontinous set of functions in E. Hence it is compact in view of Arzela-Ascoli theorem. Consequently $B : E \to E$ is a partially compact operator of E into itself.

Step V: u satisfies the inequality $u \le Au + Bu$.
By hypothesis (A_4) the equation (1.1) has a lower solution u defined on J.
Then we have

$$\begin{cases} u'(t) \le f(t, u(t)) + g\left(t, \max_{0 \le \xi \le t} u(\xi)\right), \ t \in J \\ u(0) \le u(T), \end{cases} \tag{3.19}$$

A direct application of lemma(3.3) yields that

$$u(t) \le \int_0^T G(t,s) f\left(s, \max_{0 \le \xi \le s} u(\xi)\right) ds + \int_0^T G(t,s) g\left(s, \max_{0 \le \xi \le s} u(\xi)\right) ds \tag{3.20}$$

For $t \in J$, from definitions of the operators A *and* B it follows that $u(t) \le Au(t) + Bu(t)$, for all $t \in J$ .Hence, $u \le Au + Bu$. Thus A *and* B satisfy all conditions of theorem (2.1) and we apply it to conclude that the operator equation $Ax + Bx = x$ has a solution.Consequently the integral equation and the equation (1.1) has a solution x^* defined on J.Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximation defined by (3.5) converges monotonically to x^*.
This completes the proof.

References

1.**D.D.Bainov, S. Hristova**,Differential equations with maxima, *Chapman & Hall/CRC Pure and Applied Mathematics, 2011.*

2.**S.S.Bellale**, Hybrid Fixed point theorem For theorem for Abstract Measure Differential Equation*,World Academy Of Science, Engineering and Technology,73(2013) 782-785.*

3. **S.S.Bellale, G.B.Dapke**, Approximate Solutions For Perturbed Measure Differential Equations with Maxima, *International Journal of Engineering Sciences and Research Technology, 5(9), Sep, 2016.*

4. **S.S.Bellale and G.B. Dapke,** Hybrid fixed point theorem for abstract measure integro- differential equation, *International Journal of Statistics and Applied Mathematics 2008;3(1):101-106.*

5. **S.S.Bellale and G.B. Dapke,** Existence theory for Perturbed Abstract Measure Differential Equation, *Journal of computer and mathematical Sciences, Vol.8 (11), 691-701 November 2017.*

6.**S.S.Bellale and G.B. Dapke,** Iteration Method for initial Value problems of Nonlinear second order functional differential equations, *Aryabhatta Journal of Mathematical & Informatics Vol. 10, No.2, July – Date., 2018.*

7.**S.S.Bellale , G.B.Dapke,** Existence Theorem And Extremal Solutions For Perturbed Measure Differential Equations with Maxima, *International Journal of Mathematical Archive – 7 (10), 2016, 1-11.*

8.**B.C.Dhage,** Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ. Appl. 5 (2013), 155-184.*

9.**B.C. Dhage,** Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations, *Tamkang J, Math. 45 (4) (2014), 397-426.*

10.**B.C.Dhage**, Operator theoretic techniques in the theory of nonlinear hybrid differential equations, *Nonlinar Anal.Forum 20 (2015), 15-31.*

11.**B.C.Dhage**, Nonlinear $\mathcal{D}$- set contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equation, *Malaya J. Mat. 3 (1) (2015), 62-85.*

12.**B.C.Dhage**, A new monotone iteration principle in the theory of nonlinear first order integro-differential equations, *Nonlinear Studies 22 (3) (2015, 397- 417.*

13.**B.C. Dhage**, Quadratic perturbation of periodic boundary value problems of second order ordinary differential equations, *Differ. Equ. & Appl. 2 (2010), 465-486.*

14.**B.C.Dhage, S.B.Dhage**, Approximating solutions of nonlinear first order ordinary differential equations, *GJMS Special issue for Recent Advances in Mathematical Sciences and Applications-13, GJMS Vol.2, No.2,(2014), 25-35.*

15.**B.C.Dhage, S.B.Dhage, J.R.Graef**, Dhage iteration method for initial value problems for nonlinear first order hybrid integro differential equations, *J.Fixed Point Theory Appl.17 (2016).*

16.**B.C.Dhage and S.S.Bellale**, Existence Theorem for perturbed Abstract measure differential Equations, *Nonlinear analysis*, 71(2009), e319-e328.
17.**B.C.Dhage and S.S.Bellale**, Local asymptotic Stability for nonlinear quadratic functional Integral Equations,*Electronics journal of qualitive theory of Differential equations, 10(2008) 1-23*.
18. **S.Heikkila and V.Lakshmikantham**, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, *Marcel Dekker inc., New York 1994.*
19.**J.J.Nieto, R.Rodriguez-Lopez**, Contraction mappings theorems in partially ordered sets and applications to ordinary differential equations, *Order, 22 (2005), 223-239*.
20. **D.Otrocol**, Systems of functional differential equations with maxima of mixed type, *Electron.J.Qual. Theory Differ. Equ., 2014 (2014), no.5,1-9.*
21.**A.C.M. Ran, M.C. Reurings,**A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc.Amer. Math.Soc., 132 (2004), 1435-1443*.