On Some Paranormed Zweier IS-Convergent Sequence Spaces Defined By Orlicz Function

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ABSTRACT
The main aim of this article is to introduce a new class of sequence spaces $\mathbb{z}^{IS}(M,g,q), \mathbb{z}_0^{IS}(M,g,q), \mathbb{z}_\infty^{IS}(M,g,q), m_{\mathbb{z}}^{\mathrm{IS}}(M,g,q), m_{\mathbb{z}_0}^{IS}(M,g,q)$ where $q=(q_k)$, a sequence of positive real numbers. We use Zweier transform, an Orlicz function and a bounded sequence of positive real numbers to study the above spaces. We have studied some algebraic and topological properties and also proved some inclusion relation regarding these new spaces.

Keywords : Paranorm, Ideal Statistical convergence, Zweier transform, Orlicz function.

1 Introduction
The generalization of statistical convergence was introduced as an ideal convergence by P.Kostyrko et al, Salat and Wilczynski [1]. Later on it had been studied, on paranormed Zweier ideal convergent sequence spaces defined by Orlicz function by Bipan Hazarika, Karan Tamang, B.K.Sing[11]. Motivated by this fact, the concept of Zweier ideal statistical convergent sequence spaces that are defined by Orlicz function is introduced in this paper . In this paper the Ideal statistical convergent is denoted by IS convergent . Throughout the paper W denote the vector space of all sequences.
Sengonul[2] introduced the Zweier sequence spaces $\mathbb{z}$ and $\mathbb{z}_0$ as follows
$\mathbb{z}=\{x=(x_k)\in \mathrm{W}: Z^q x\in c\}$ and $\mathbb{z}_0=\{x=(x_k)\in \mathrm{W}: Z^q x\in c_0\}$

2 Preliminaries
Definition 2.1.[17,19] Let X be a linear space, A function $g: X\to\mathbb{R}$ is called paranorm if
- $g(x)\geq 0$ for all $x\in X$.
- $g(-x)=g(x)$, for all $x\in X$.
- $g(x_1+x_2)\leq g(x_1)+g(x_2)$, for all $x_1,x_2\in X$.
- If (λ_n) is a sequence of scalars with $\lambda_n\to\lambda_0$ as $n\to\infty$ and $x_n,a\in X$ with $x_n\to a$ as $n\to\infty$ in the sense that $g(x_n-a)\to 0$ as $n\to\infty$, then $\lambda_n x_n\to\lambda_0 a$ as $n\to\infty$, in the sense that $g(\lambda_n x_n-\lambda_0 a)\to 0$ as $n\to\infty$.

  A paranorm g for which $g(x)=0$ implies $x=0$ is called total paranorm and the pair (X,g) is called a total paranormed space.

Definition 2.2. Let X be a non empty set. Then a family of set $I\subseteq 2^X$ (power sets of X) is said to be an ideal on X if and only if
- $\emptyset\in I$.
- I is additive, i.e. $A,B\in I\Rightarrow A\cup B\in I$.
- I is hereditary, i.e. $A\in I, B\subset A\Rightarrow B\in I$.

Definition 2.3. An ideal $I\subseteq 2^X$ is called non trivial if $I\neq 2^X$.

Definition 2.4. A non trivial ideal $I\subset 2^X$ is called admissible if $I\supseteq\{\{x\}: x\in X\}$.

Definition 2.5.[16] Let $I \subseteq 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence (x_k) is said to be I- Statistically Convergent to ζ in (X,g), if for every $\epsilon > 0$ and every $\delta > 0$, $\left\{n \in \mathbb{N}: \frac{1}{n}|\{k \le n: g(x_k - \zeta) \ge \varepsilon\}| \ge \delta\right\} \in I$. ζ is called (g,I) - statistical limit of the sequence (x_k) and we write $(g,I) - st\ lim x_k = \zeta$.

Definition 2.6. [16] A sequence $x = (x_k)$ is said to be I- Statistically cauchy in (X,g), if for every $\epsilon > 0$ and every $\delta > 0$, there exists a number $N = N(\varepsilon)$ such that , $\left\{n \in \mathbb{N}: \frac{1}{n}|\{k \le n: g(x_k - x_N) \ge \varepsilon\}| \ge \delta\right\} \in I$.

Definition 2.7. A function $M: [0,\infty) \to [0,\infty)$ is said to be an Orlicz function , which is continuous, non decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.
Nakano[17] and Musielak[18] and others have defined and discussed a modular function that replaces the convexity of Orlicz function M with $M(x+y) \le M(x) + M(y)$.

Remark 2.8.

1. If M is an Orlicz function, then $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.
2. If a constant $k > 0$ exists, an Orlicz function M can be said to satisfy Δ_2- condition for all values of u if $M(Lu) \le kLM(u)$ for all values of $L > 1$[3].
3. The sequence space $l_M = \left\{x \in W: \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}\right) < \infty$ for some $\rho > 0\}$ was constructed by Lindenstrauss and Tzafriri [4] used the idea of Orlicz function.
4. The space l_M becomes a Banach space with the norm $||x|| = \inf\{\rho > 0: \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \le 1\}$. Which is called an Orlicz sequence space.
5. The spaces l_M and l_p are closely related as each is an Orlicz sequence space with $M(t) = t^p$ for $1 \le p < \infty$.

Later on Orlicz sequence spaces were investigated by Parashar and Chaudhary [5], Esi [6], Tripathy et al [7], Bhardwaj and Singh [8], Et[9] Esi and Et[10], Hazarika et al [11] and many others.

- **Definition:2.9.** A sequence space S is said to be solid (or normal) if $(\alpha_k x_k) \in S$ Whenever $(x_k) \in S$ and for all sequence (α_k) of scalars with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$.
- Let $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$ and let S be a sequence space. A K- step space of S is a sequence space $\lambda_K^S = \{(x_{k_n}) \in W : (k_n) \in S\}$

Definition 2.10.[13] Let (x_k) and (y_k) be two sequences. We say that $x_k = y_k$ for almost all k relative to I (a.a.k.r.I), if $\{k \in \mathbb{N}: x_k \ne y_k\} \in I$.

Lemma 2.11 .[15] Let $k \in F(I)$ and $M \subseteq N$. If $M \notin I$ then $M \cap K \notin I$.

Lemma 2.12.[13] The sequence space S is solid implies that S is monotone.

3 Main Result

Throughout the article $\mathbb{z}^{IS}, \mathbb{z}_0^{IS}, \mathbb{z}_\infty^{IS}$ represents Zweier IS- convergent, Zweier IS-null, Zweier IS bounded sequence spaces respectively.

In this article, we introduce following class of sequence spaces:

$\mathbb{z}^{IS}(M,g,q) = \left\{x = (x_k): \{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \le n: \left[M\left(\frac{g(Z^q x)_n - L}{\rho}\right)\right]^{q_n} \ge \varepsilon\right\}\right| \ge \delta\right\} \in I$ for some $\rho > 0, L \in \mathbb{C}\}$.

$\mathbb{z}_0^{IS}(M,g,q) = \{x = (x_k): \{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \le n: \left[M\left(\frac{g(Z^q x)_n}{\rho}\right)\right]^{q_n} \ge \varepsilon\right\}\right| \ge \delta\} \in I$ for some $\rho > 0, L \in \mathbb{C}\}$.

$\mathbb{z}_\infty^{IS}(M,g,q) = \left\{x = (x_k): \left\{n \in \mathbb{N}: \frac{1}{n}\left|\{k \le n: [M\left(\frac{g(Z^q x)_n}{\rho}\right)^{q_n} \ge k\}\right| \ge \delta\right\} \in I$ for some $k > 0$.

Also we write $m_{\mathbb{z}}^{IS}(M,g,q) = \mathbb{z}^{IS}(M,g,q) \cap \mathbb{z}_\infty^{IS}(M,g,q)$ and
$m_{\mathbb{z}_0}^{IS}(M,g,q) = \mathbb{z}_0^{IS}(M,g,q) \cap \mathbb{z}_\infty^{IS}(M,g,q)$, where $q = (q_k)$ is a sequence of positive real numbers.

$l_\infty(M,g,q) = \{x = (x_k) \in W: \frac{1}{n}|sup\left[M\left(\frac{g(Z^q x)_n}{\rho}\right)\right]^{q_n}| < \infty\}$.

Theorem 3.1. The class of sequences $\mathfrak{z}^{IS}(M,g,q), \mathfrak{z}_0^{IS}(M,g,q), m_{\mathfrak{z}}^{IS}(M,g,q)$ *and* $m_{\mathfrak{z}_0}^{IS}(M,g,q)$ are linear spaces.

Proof. We shall prove the result for the space $\mathfrak{z}^{IS}(M,g,q)$. Let $x=(x_k), y=(y_k) \in \mathfrak{z}^{IS}(M,g,q)$ and let α, β be scalars. For given $\varepsilon > 0$ and every $\delta > 0$. We have

$$A_1 = \left\{x=(x_k): \left\{n \in \mathbb{N}: \frac{1}{n}\left|\{k \le n: [M\left(\frac{g((Z^q x)_n - L_1)}{\rho_1}\right)]^{q_n} \ge \frac{\varepsilon}{2}\right\}\right| \ge \delta\right\} \in I \text{ for some } L_1 \in \mathbb{C}\}.$$
$$A_2 = \left\{y=(y_k): \left\{n \in \mathbb{N}: \frac{1}{n}\left|\{k \le n: [M\left(\frac{g((Z^q y)_n - L_2)}{\rho_2}\right)]^{q_n} \ge \frac{\varepsilon}{2}\right\}\right| \ge \delta\right\} \in I \text{ for some } L_2 \in \mathbb{C}\}.$$

Let $\rho = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non decreasing and convex function, we have

$$[M\left(\frac{g(\alpha(Z^q x)_n + \beta(Z^q y)_n) - (\alpha L_1 + \beta L_2)}{\rho}\right)]^{q_n} \le \left[M\left(\frac{g(\alpha((Z^q x)_n - L_1))}{\rho}\right)\right]^{q_n} + \left[M\left(\frac{g(\beta((Z^q y)_n - L_2))}{\rho}\right)\right]^{q_n}$$
$$\le \left[M\left(\frac{g((Z^q x)_n - L_1)}{\rho_1}\right)\right]^{q_n} + \left[M\left(\frac{g((Z^q y)_n - L_2)}{\rho_2}\right)\right]^{q_n}$$
$$\left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \le n: \left[M\left(\frac{g(\alpha(Z^q x)_n + \beta(Z^q y)_n) - (\alpha L_1 + \beta L_2)}{\rho}\right)\right]^{q_n} \ge \varepsilon\right\}\right| \ge \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \le n: \left[M\left(\frac{g((Z^q x)_n - L_1)}{\rho_1}\right)\right]^{q_n} \ge \frac{\varepsilon}{2}\right\}\right| \ge \delta\right\}$$
$$\cup \left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \le n: \left[M\left(\frac{g((Z^q y)_n - L_2)}{\rho_2}\right)\right]^{q_n} \ge \frac{\varepsilon}{2}\right\}\right| \ge \delta\right\} \subseteq A_1 \cup A_2 \in I.$$

Therefore $(\alpha(Z^q x)_n + \beta(Z^q y)_n) \in \mathfrak{z}^{IS}(M,g,q)$. Hence $\mathfrak{z}^{IS}(M,g,q)$ is a linear space. The proof for other spaces will proceed in the same manner.

Theorem 3.2. The spaces $m_{\mathfrak{z}}^{IS}(M,g,q)$ and $m_{\mathfrak{z}_0}^{IS}(M,g,q)$ are paranormed spaces, with the paranorm $g'(x) = \inf\{\rho^{\frac{q_n}{H}}: \frac{1}{n}|Sup_n M\left(\frac{g((Z^q x)_n)}{\rho}\right)| \le 1, \text{ for some } \rho > 0\}$, where $H = \max\{1, Sup_n q_n\}$.

Proof: Clearly $g'(-x) = g'(x)$ and $g'(\theta) = 0$. Let $x=(x_k)$ and $y=(y_k)$ be two elements in $m_{\mathfrak{z}}^{IS}(M,g,q)$. Now for $\rho_1, \rho_2 > 0$, we denote $A_1 = \{\rho_1: \frac{1}{n}|Sup_n M\left(\frac{g((Z^q x)_n)}{\rho_1}\right)| \le 1\}$, and $A_2 = \{\rho_2: \frac{1}{n}|Sup_n M\left(\frac{g((Z^q y)_n)}{\rho_2}\right)| \le 1\}$. Let us take $\rho = \rho_1 + \rho_2$. By the convexity of Orlicz functions M, we get $M\left(\frac{g(Z^q(x+y)_n)}{\rho}\right) \le \frac{\rho_1}{\rho_1+\rho_2} M\left(\frac{g((Z^q x)_n)}{\rho_1}\right) + \frac{\rho_1}{\rho_1+\rho_2} M\left(\frac{g((Z^q y)_n)}{\rho_2}\right)$ which in terms give us $\frac{1}{n}|Sup_n M\left(\frac{g((Z^q(x+y)_n)}{\rho}\right)| \le 1$ and
$g'(x+y) = \inf\left\{(\rho_1+\rho_2)^{\frac{q_n}{H}}: \rho_1 \in A_1, \rho_2 \in A_2\right\} \le \inf\left\{\rho_1^{\frac{q_n}{H}}: \rho_1 \in A_1\right\} + \inf\left\{\rho_2^{\frac{q_n}{H}}: \rho_2 \in A_2\right\} = g'(x) + g'(y)$.
Let $t^m \to L$, where $t^m, L \in \mathbb{C}$ and let $g'(x^m - x) \to 0$ *as* $m \to \infty$.
To prove that $g'(t^m x^m - Lx) \to 0$ *as* $m \to \infty$, we put
$A_3 = \{\rho_m > 0: \frac{1}{n}|Sup_n[M\left(\frac{g((Z^q x^m)_n)}{\rho_m}\right)]^{q_n}| \le 1\}$, *and*
$A_4 = \{\rho_s > 0: \frac{1}{n}|Sup_n[M\left(\frac{g((Z^q(x^m - x))_n)}{\rho_s}\right)]^{q_n}| \le 1\}$. By the continuity of M, we observe that

$$M\left(\frac{g(Z^q(t^m x^m - Lx)_n)}{g(t^m - L)\rho_m + g(L)\rho_s}\right) \le M\left(\frac{g(Z^q(t^m x^m - Lx^m)_n)}{g(t^m - L)\rho_m + g(L)\rho_s}\right) + M\left(\frac{g(Z^q(Lx^m - Lx)_n)}{g(t^m - L)\rho_m + g(L)\rho_s}\right)$$
$$\le \left(\frac{g(t^m - L)_n \rho_m}{g(t^m - L)\rho_m + g(L)\rho_s}\right) M\left(\frac{g((Z^q x^m)_n)}{\rho_m}\right) + \left(\frac{g(L)\rho_s}{g(t^m - L)\rho_m + g(L)\rho_s}\right) M\left(\frac{g((Z^q x^m - x)_n)}{\rho_s}\right)$$

$\frac{1}{n}\left|Sup\left[M\left(\frac{g(Z^q(t^m x^m - Lx)_n)}{g(t^m - L)\rho_m + g(L)\rho_s}\right)\right]\right| \le 1$ and consequently,

$$g'(t^m x^m - Lx) = \inf\left\{\left[(g(t^m - L))\rho_m + g(L)\rho_s\right]^{\frac{q_n}{H}}; \rho_m \in A_3, \rho_s \in A_4\right\}$$
$$\le \max\left\{1, g(t^m - L)^{\frac{q_n}{H}}\right\} g'(x^m) + \max\left\{1, (g(L))^{\frac{q_n}{H}}\right\} g'(x^m - x).$$

Note that $g'(x^m) \le g'(x) + g'(x^m - x)$ for all $m \in \mathbb{N}$. Hence the right hand side of the above relation tends to zero as $m \to \infty$. This completes the proof.

Theorem 3.3. $m_{\mathfrak{z}}^{IS}(M,g,q)$ is a closed subspace of $l_\infty(M,g,q)$.

Proof. Let $(x_k^{(i)})$ be a Cauchy sequence in $m_{\mathfrak{z}}^{IS}(M,g,q)$ such that $x^{(i)} \to x$. We show that $x \in m_{\mathfrak{z}}^{IS}(M,g,q)$. Since $(x_k^{(i)}) \in m_{\mathfrak{z}}^{IS}(M,g,q)$, then there exists a_i such that $\left\{n \in \mathbb{N}: \frac{1}{n}\left|\left\{k \le n: \left[M\left(\frac{g\left((Z^q x_k^i)_n - a_i\right)}{\rho}\right)\right]^{q_n} \ge \varepsilon\right\}\right| \ge \delta\right\} \in I$. We need to show that

1. (a_i) converges to a

2. If $U = \left\{x = x_k : \left\{n \in \mathbb{N} : \frac{1}{n} |k \leq n : \left[M\left(\frac{g((Z^q x)_n - L)}{\rho}\right)\right]^{q_n} < \varepsilon\right\} \right| \geq \delta\right\}$, then $U^c \in I$.

(1) Since $(x_k^{(i)})$ is a Cauchy sequence in $m_{\mathbb{z}}^{\mathrm{IS}}(M, g, q)$ then for a given $\varepsilon > 0$ and every $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that $\frac{1}{n}\left|\sup\left[M\left(\frac{g((Z^q x_k^i)_n - (Z^q x_k^j)_n)}{\rho}\right)\right]^{q_n} < \frac{\varepsilon}{3}\right| \geq \delta$, for all $i, j \geq k_0$. For a given $\varepsilon > 0$ and every $\delta > 0$, we have

$$B_{ij} = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \left[M\left(\frac{g\left(\left(Z^q x_k^i\right)_n - \left(Z^q x_k^j\right)_n\right)}{\rho}\right)\right]^{q_n} < \frac{\varepsilon}{3}\right.\right\}| \geq \delta\}$$

$$B_i = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \left[M\left(\frac{g\left(\left(Z^q x_k^i\right) - a_i\right)_n}{\rho}\right)\right]^{q_n} < \frac{\varepsilon}{3}\right.\right\}| \geq \delta\}$$

$$B_j = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \left[M\left(\frac{g\left(\left(Z^q x_k^j\right) - a_j\right)_n}{\rho}\right)\right]^{q_n} < \frac{\varepsilon}{3}\right.\right\}| \geq \delta\}$$

Then $B_{ij}^c, B_i^c, B_j^c \in I$. Let $B^c = B_{ij}^c \cup B_i^c \cup B_j^c$, where $B = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left[M\left(\frac{g(a_i - a_j)}{\rho}\right)\right]^{q_n} < \varepsilon\right\}\right| \geq \delta\right\}$.
Then $B^c \in I$ we choose $n_0 \in B^c$, then for each $i, j \geq n_0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left[M\left(\frac{g(a_i - a_j)}{\rho}\right)\right]^{q_n} < \varepsilon\right\}\right| \geq \delta\right\}$$

$$\supseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|k \leq n : \left[M\left(\frac{g\left(\left(Z^q x_k^i\right) - a_i\right)_n}{\rho}\right)\right]^{q_n} < \frac{\varepsilon}{3}\right.\right\}| \geq \delta\}$$

$$\cap \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \left[M\left(\frac{g\left(\left(Z^q x_k^i\right)_n - \left(Z^q x_k^j\right)_n\right)}{\rho}\right)\right]^{q_n} < \frac{\varepsilon}{3}\right.\right\}| \geq \delta\}$$

$$\cap \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \left[M\left(\frac{g\left(\left(Z^q x_k^j\right) - a_j\right)_n}{\rho}\right)\right]^{q_n} < \frac{\varepsilon}{3}\right.\right\}| \geq \delta\}.$$

Then $(a)_j$ is a Cauchy sequence of scalars in $\mathbb{C}$ and so there exists a scalar $a \in \mathbb{C}$ such that $a_j \to a \; as \; j \to \infty$.

(2) Let $0 < \zeta < 1$ be given. We show that if $U = \left\{x = (x_k) : \left\{n \in \mathbb{N} : \frac{1}{n} |k \leq n : \left[M\left(\frac{g((Z^q x_k) - a)_n}{\rho}\right)\right]^{q_n} < \zeta\right\} \right| \geq \delta\right\}$, then $U^c \in I$. Since $x^{(i)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$P = \{n \in \mathbb{N} : \frac{1}{n} | \left\{k \leq n : \left[M\left(\frac{g\left(\left(Z^q x_k^{q_0}\right)_n - (Z^q x)_n\right)}{\rho}\right)\right]^{q_n} < \left(\frac{\zeta}{3D}\right)^H | \geq \delta\right\}$ – – – – – – (1) which implies $P^c \in I$.

The number q_0 can be so chosen that together with equation (1), we have

$Q = \{n \in \mathbb{N} : \frac{1}{n} | \left\{k \leq n : \left[M\left(\frac{g(a_{q_0} - a)}{\rho}\right)\right]^{q_n} < \left(\frac{\zeta}{3D}\right)^H | \geq \delta\right\}$. Then we have $Q^c \in I$.

Since $\{n \in \mathbb{N} : \frac{1}{n} | \left\{k \leq n : \left[M\left(\frac{g\left(Z^q x_k^{q_0} - a_{q_0}\right)_n}{\rho}\right)\right]^{q_n} > \zeta | \geq \delta\right\} \in I$. Then we have a subset $S \in \mathbb{N}$ such that $S^c \in I$, where $S = \{n \in \mathbb{N} : \frac{1}{n} | \left\{k \leq n : \left[M\left(\frac{g\left(Z^q x_k^{q_0} - a_{q_0}\right)_n}{\rho}\right)\right]^{q_n} < \left(\frac{\zeta}{3D}\right)^H | \geq \delta\right\}$. Let $U^c = P^c \cup Q^c \cup S^c$ where $U = \left\{x = (x_k) : \left\{n \in \mathbb{N} : \frac{1}{n} |k \leq n : \left[M\left(\frac{g((Z^q x_k - a)_n}{\rho}\right)\right]^{q_n} < \zeta\right\} \right| \geq \delta\right\}$. Therefore

$$\left\{n \in \mathbb{N} : \frac{1}{n} |k \leq n : \left[M\left(\frac{g((Z^q x_k - a)_n}{\rho}\right)\right]^{q_n} < \zeta\right\} | \geq \delta\}$$

$$\supseteq \left\{n \in \mathbb{N} : \frac{1}{n} |k \leq n : \left[M\left(\frac{g(\left(Z^q x_k^{q_0}\right)_n - \left(Z^q x_k\right)_n}{\rho}\right)\right]^{q_n} < \left(\frac{\zeta}{3D}\right)^H\right\} | \geq \delta\}$$

$$\cap \left\{n \in \mathbb{N} : \frac{1}{n} |k \leq n : \left[M\left(\frac{g(\left(Z^q x_k^{q_0} - a_{q_0}\right)_n}{\rho}\right)\right]^{q_n} < \left(\frac{\zeta}{3D}\right)^H\right\} | \geq \delta\}$$

$$\cap \left\{n \in \mathbb{N} : \frac{1}{n} |k \leq n : \left[M\left(\frac{g(a_{q_0} - a)}{\rho}\right)\right]^{q_n} < \left(\frac{\zeta}{3D}\right)^H\right\} | \geq \delta\}.$$

Then the result follows. Since the inclusions $m_{\mathbb{z}}^{\mathrm{IS}}(M, g, q) \subset l_\infty(M, g, q)$ and $m_{\mathbb{z}_0}^{\mathrm{IS}}(M, g, q) \subset l_\infty(M, g, q)$ are strict, then the following results based on theorem 3.3.

Theorem 3.4. The spaces $m_{\mathbb{z}}^{\mathrm{IS}}(M,g,q)$ and $m_{\mathbb{z}_0}^{\mathrm{IS}}(M,g,q)$ are nowhere dense subsets of $l_\infty(M,g,q)$.

Theorem 3.5. The spaces $m_{\mathbb{z}}^{\mathrm{IS}}(M,g,q)$ and $m_{\mathbb{z}_0}^{\mathrm{IS}}(M,g,q)$ are not separable.

Proof. Let $H=\{h_1<h_2<h_3<\cdots\}$ be an infinite subset of $\mathbb{N}$ that $H\in I$, $Q_n=\begin{cases} n+1 & if\ n\in H \\ 1 & otherwise\end{cases}$.
Let $Q_0=\{(x_n): x_n=0\ or\ 1, n\in H\quad and\quad x_n=0\ , otherwise\}$. Since H is infinite, so Q_0 is uncountable. Consider the class of open balls $B_1=\left\{B\left(Z,\frac{1}{2}\right): Z\in Q_0\right\}$. Let C_1 be an open cover of $m_{\mathbb{z}_0}^{\mathrm{IS}}(M,g,q)$ and $m_{\mathbb{z}}^{\mathrm{IS}}(M,g,q)$ containing B_1. Since B_1 is uncountable, so C_1 cannot be reduced to a countable sub cover for $m_{\mathbb{z}_0}^{\mathrm{IS}}(M,g,q)$ as well as $m_{\mathbb{z}}^{\mathrm{IS}}(M,g,q)$. Thus $m_{\mathbb{z}}^{\mathrm{IS}}(M,g,q)$ and $m_{\mathbb{z}_0}^{\mathrm{IS}}(M,g,q)$ are not separable.

Theorem 3.6. Let $H=Sup_n q_n<\infty$ and I an admissible ideal. Then the following are equivalent

(1) $x=(x_k)\in \mathbb{z}^{IS}(M,g,q)$
(2) there exists $y=(y_k)\in \mathbb{z}(M,g,q)$ such that $x_k=y_k$ for a.a.k.r.I.
(3) there exists $y=(y_k)\in \mathbb{z}(M,g,q)$ and $z=(z_k)\in \mathbb{z}_0^{IS}(M,g,q)$ such that $x_k=y_k+z_k$ for all $k\in\mathbb{N}$ and $\left\{y=(y_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\le n: [M\left(\frac{g((Z^q y)_n-L)}{\rho}\right)]^{q_n}\ge\varepsilon\right\}\right|\ge\delta\right\}\in I$
(4) there exists a subset $K=\{k_1<k_2<k_3<\cdots\}$ of $\mathbb{N}$ such that $K\in\mathcal{F}(I)$ and $\lim_n \frac{1}{n}[M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)]^{q_n}=0$

Proof. (1)$\Longrightarrow$(2)
Let $x=(x_k)\in \mathbb{z}^{IS}(M,g,q)$. Then for a given $\varepsilon>0$ and every $\delta>0$, there exists $L\in\mathbb{C}$ such that $\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\le n: [M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)]^{q_n}\ge\varepsilon\right\}\right|\ge\delta\right\}\in I$. Let (p_j) be an increasing sequence with $p_j\in\mathbb{N}$, such that $\left\{x=(x_k):\left\{n\le p_j:\frac{1}{n}\left|\{k\le n: [M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)]^{q_n}\ge j^{-1}\right\}\right|\ge\delta\right\}\in I$. Define a sequence $y=(y_k)$ as $y_k=x_k$, for all $k\le p_1$. For $p_j<k\le p_{j+1}$, $j\in\mathbb{N}$, $y_k=\begin{cases} x_k & for\ \frac{1}{n}\left|\left[M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)\right]^{q_n}<j^{-1}\right|\ge\delta \\ L & otherwise\end{cases}$. Then $(y_k)\in \mathbb{z}(M,g,q)$ and from the following inclusion $\{k\le p_j: x_k\ne y_k\}\subseteq\left\{k\le p_j:\frac{1}{n}\left|\left[M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)\right]^{q_n}\ge\varepsilon\right\}\right|\ge\delta\}\in I$. We get $x_k=y_k$ for a.a.k.r.I.

(2)$\Longrightarrow$(3)
For $(x_k)\in \mathbb{z}^{IS}(M,g,q)$, then there exists $(y_k)\in \mathbb{z}(M,g,q)$ such that $x_k=y_k$ for a.a.k.r.I. Let $\{k\in\mathbb{N}: x_k\ne y_k\}$, then $k\in I$. Defined a sequence (z_k) as $z_k=\begin{cases} x_k-y_k, & if\ k\in K \\ 0 & otherwise\end{cases}$. Then $(z_k)\in \mathbb{z}_0^{IS}(M,g,q)$ and $(y_k)\in \mathbb{z}(M,g,q)$.

(3)$\Longrightarrow$(4)
Suppose (3) holds, Let be given $\varepsilon>0$ and every $\delta>0$.
Let $A_1=\{z=(z_k):\{n\in\mathbb{N}:\frac{1}{n}|\left\{k\le n:\left[M\left(\frac{g(Z^q z)_n}{\rho}\right)\right]^{q_n}\ge\varepsilon\right\}|\ge\delta\}\in I\}$ and K$=A_1^c=\{k_1<k_2<k_3<\cdots\}\in F(I)$.
Then we have $\lim_n \frac{1}{n}[M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)]^{q_n}=0$.

(4)$\Longrightarrow$(1)
Let $K=\{k_1<k_2<k_3<\cdots\}\in F(I)$ and $\lim_n \frac{1}{n}[M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)]^{q_n}=0$.
Then for any $\varepsilon>0$ and by lemma 2.13,
we have $\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\le n:\left[M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)\right]^{q_n}\ge\varepsilon\right\}\right|\ge\delta\right\}$

$$\subseteq\ K^c\cup\left\{x=(x_k):\left\{n\in K:\frac{1}{n}\left|\{k\le n: [M\left(\frac{g((Z^q x)_n-L)}{\rho}\right)]^{q_n}\ge\varepsilon\right\}\right|\ge\delta\right\}\in I.$$

Thus $(x_k)\in \mathbb{z}^{IS}(M,g,q)$.

Theorem 3.7

Let $f=inf_n q_n$ and $H=sup_n q_n$, then the following results are equivalent.
1. $H<\infty$ and $f>0$
2. $\mathbb{z}_0^{IS}(M,g,q)=\mathbb{z}_0^I, where\ \mathbb{z}_0^I=\{(x_k)\in W:\{k\in\mathbb{N}; I-\lim Z^q x=0\}\in I\}$

Proof. Suppose that $H<\infty$ and $f>0$, then the inequalities $\min\{1,s^f\}\le s^{q_n}\le\max\{1,s^H\}$ hold for any $s>0$ and for all $n\in\mathbb{N}$. Therefore the equivalence of (1) and (2) is obvious.

Theorem 3.8

Let M_1 and M_2 be two Orlicz functions that satisfy the Δ_2 – condition, then
1. $W(M_2,g,q)\subseteq W(M_1M_2,g,q)$

2. $W(M_1,g,q)\cap W(M_2,g,q)\subseteq W(M_1+M_2,g,q)$ where $W=\mathfrak{z}^{IS},\mathfrak{z}_0^{IS},m_{\mathfrak{z}}^{\mathrm{IS}},m_{\mathfrak{z}_0}^{IS}$

Proof. (1) Let $x=(x_k)\in\ \mathfrak{z}^{IS}(M_2,g,q)$. Let given $\varepsilon>0$ and every $\delta>0$. For some $\rho>0$, we have

$\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:[M_2\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\right\}\in I\ ----(1)$. Let $\varepsilon>0$ and choose $0<\eta<1$ such that $M_1(i)\leq\varepsilon$ for $0\leq i\leq\eta$, we define $t_n=\frac{1}{n}[M_2\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]$ and consider

$\lim_{n\in\mathbb{N}:0\leq t_n\leq\eta}[M_1(t_n)]^{q_n}=\lim_{n\in\mathbb{N}:t_n\leq\eta}[M_1(t_n)]^{q_n}+\lim_{n\in\mathbb{N}:t_n>\eta}[M_1(t_n)]^{q_n}$ we have

$\lim_{n\in\mathbb{N}:t_n\leq\eta}[M_1(t_n)]^{q_n}\leq[M_1(k)]^H+\lim_{n\in\mathbb{N}:t_n\leq\eta}(t_n)^{q_n},\mathrm{k=constant},H=\sup q_n\ ----(2)$. The procedure for second term ((ie) $t_n>\eta$), is as follows $t_n<\frac{t_n}{\eta}<1+\frac{t_n}{\eta}$. Since M_1 is non decreasing and convex, it follows that $M_1(t_n)<M_1\left(1+\frac{t_n}{\eta}\right)\leq\frac{1}{2}M_1(k)+\frac{1}{2}M_1\left(\frac{kt_n}{\eta}\right)$. Since M_1satisfies Δ_2- conditions,
we can write $M_1(t_n)<\ \frac{1}{2}K\frac{t_n}{\eta}M_1(k)+\frac{1}{2}K\left(\frac{t_n}{\eta}\right)M_1(k)=K\frac{t_n}{\eta}M_1(k)$.We were given the following estimates:
$\lim_{n\in\mathbb{N}:t_n>\eta}[M_1(t_n)]^{q_n}\leq\max\{1,(K\eta^{-1}M_1(k))^H\}\lim_{n\in\mathbb{N}:t_n>\eta}[t_n]^{q_n}\ ----(3)$
From (1)-(3), it follows that
$\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:[M_1M_2\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$
Hence $\mathfrak{z}^{IS}(M_2,g,q)\subseteq\mathfrak{z}^{IS}(M_1M_2,g,\mathrm{q})$.
(2) Let $(x_k)\in\mathfrak{z}^{IS}(M_1,g,q)\cap\ \mathfrak{z}^{IS}(M_2,g,q)$. Let be given $\varepsilon>0$ and every $\delta>0$. Then there exists $\rho>0$ such that
$\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:[M_1\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\right\}\in I$ and
$\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:[M_2\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$
The following relations to the remainder of the proof

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|[(M_1+M_2)\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\}$$
$$\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\left|[M_1\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\}\cup\left\{n\in\mathbb{N}:\frac{1}{n}\left|[M_2\left(\frac{g((Z^qx)_n-L)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\}.$$

Taking $M_2(x)=x\ and\ M_1(x)=M(x)$ for all $x\in[0,\infty)$. We have the following result.

Corollary 3.9. $W\subseteq W(M,g,q)$ where $W=\mathfrak{z}^{IS},\mathfrak{z}_0^{IS},m_{\mathfrak{z}}^{\mathrm{IS}},m_{\mathfrak{z}_0}^{IS}$

Theorem 3.10. The spaces $\mathfrak{z}_0^{IS}(M,g,q)$ and $m_{\mathfrak{z}_0}^{\mathrm{IS}}(M,g,q)$ are solid and monotone.

Proof. Let $(x_k)\in\ \mathfrak{z}_0^{IS}(M,g,q)$. Then for a given $\varepsilon>0$ and then there exists $\delta>0,\rho>0$ such that
$\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:[M\left(\frac{g((Z^qx)_n)}{\rho}\right)]^{q_n}\geq\varepsilon\right\}\right|\geq\delta\right\}\in I\ -----$ (1). Let (α_k) be a sequence of scalars with $|\alpha_k|\leq1$, for all $k\in\mathbb{N}$. Since $|\alpha_k|^{q_k}\leq\max\{1,|\alpha_k|^H\}\leq1$, for all $k\in\mathbb{N}$, where $H=\sup q_k\geq0$. Then the result follows from the equation (1), and the following inequality

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:\left[M\left(\frac{g(\alpha_k(Z^qx)_n)}{\rho}\right)\right]^{q_n}\geq\varepsilon\right\}\right|\leq\frac{1}{n}|\alpha_k|^{q_n}\left|k\leq n:\ \left[M\left(\frac{g((Z^qx)_n)}{\rho}\right)\right]^{q_n}\geq\varepsilon\right|$$
$$\leq\frac{1}{n}|\{k\leq n:\left[M\left(\frac{g((Z^qx)_n)}{\rho}\right)\right]^{q_n}\geq\varepsilon|.$$

By lemma 2.12, the space $\mathfrak{z}_0^{IS}(M,g,q)$ is monotone.

Result 3.11. The spaces $\mathfrak{z}^{IS}(M,g,q)$ and $m_{\mathfrak{z}}^{\mathrm{IS}}(M,g,q)$ are neither monotone nor solid.

Example 3.12, Let $I=I_\delta$ and $M(x)=x^3$ for all $x\in[0,\infty)$. Consider the K-step space $W_k(M,g,q)$ of $W(M,g,q)$ defined as follows. Let $(x_k)\in W(M,g,q)$ and let $(y_k)\in\ W_k(M,g,q)$ be such that $y_k=\begin{cases}x_k\ if\ k\ is\ even\\0\ otherwise\end{cases}$. Consider the sequence x_k defined by $x_k=1$ for all $k\in\mathbb{N}$. Then $(x_k)\in\ \mathfrak{z}^{IS}(M,g,q)$ but K- step space preimage does not belongs to $\mathfrak{z}^{IS}(M,g,q)$. Thus $\mathfrak{z}^{IS}(M,g,q)$ is not monotone. Hence $\mathfrak{z}^{IS}(M,g,q)$ is not solid.

Theorem 3.13. Let $q=(q_n)$ and $r=(r_n)$ be two sequences of positive real numbers. Then $m_{\mathfrak{z}_0}^{\mathrm{IS}}(M,g,q)\supseteq m_{\mathfrak{z}_0}^{\mathrm{IS}}(M,g,r)$ if and only if $\lim_{n\in k}inf\frac{q_n}{r_n}>0$, where $K\subseteq\mathbb{N}$ such that $K^c\in I$.

Proof. Let $\lim_{n\in k}inf\frac{q_n}{r_n}>0$ and $(x_n)\in m_{\mathfrak{z}_0}^{\mathrm{IS}}(M,g,r)$. Then there exist $\beta>0$ such that $q_n>\beta r_n$, for all sufficiently large $n\in K$. Let $(x_k)\in m_{\mathfrak{z}_0}^{\mathrm{IS}}(M,g,r)$ then for given $\varepsilon>0$ and $\delta>0$, we have

$B=\left\{x=(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:[M\left(\frac{g((Z^q x)_n)}{\rho}\right)]^{r_n}\ge\varepsilon\right\}\right|\ge\delta\right\}\in I$. Let $G_0=K^c\cup B$. Then we get $G_0\in I$.

Then for all sufficiently large $n\in G_0$,

$\left\{(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:[M\left(\frac{g((Z^q x)_n)}{\rho}\right)]^{q_n}\ge\varepsilon\right\}\right|\ge\delta\right\}\subseteq\left\{(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:\left[M\left(\frac{g((Z^q x)_n)}{\rho}\right)\right]^{\beta r_n}\ge\varepsilon\right\}\right|\ge\delta\right\}\in I$.

(ie) $\left\{(x_k):\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:[M\left(\frac{g((Z^q x)_n)}{\rho}\right)]^{q_n}\ge\varepsilon\right\}\right|\ge\delta\right\}\in I$. Therefore $(x_k)\in m_{z_0}^{IS}(M,g,p)$. The converse part of the result obviously.

Theorem 3.14. Let $q=(q_n)$ and $r=(r_n)$ be two sequences of positive real numbers. Then $m_{z_0}^{IS}(M,g,r)\supseteq m_{z_0}^{IS}(M,g,q)$

if and only if $\lim_{n\in k} inf\frac{r_n}{q_n}>0$, where $K\subseteq\mathbb{N}$ such that $K^c\in I$.

Proof. The proof follows similarly as the proof of theorem 3.13

Theorem 3.15. Let $q=(q_n)$ and $r=(r_n)$ be two sequences of positive real numbers. Then $m_{z_0}^{IS}(M,g,r)=m_{z_0}^{IS}(M,g,q)$ if and only if $\lim_{n\in k} inf\frac{q_n}{r_n}>0,\ and\ \lim_{n\in k} inf\frac{r_n}{q_n}>0$, where $K\subseteq\mathbb{N}$ such that $K^c\in I$.

Proof. By combining theorem 3.13 and theorem 3.14, we get the required result.

4 Conclusions

In this study of constructing a new sequence spaces defined by Orlicz function has been employed by several authors. Although B.Hazarika, K.Tamang,B.K.Sing defined by "Paranormed Zweier Ideal convergent sequence spaces defined by Orlicz function was studied earlier. In this paper we introduced paranormed Zweier IS – Convergent sequence spaces defined by Orlicz function. Further these spaces can be developed for double sequences using Modulus function.

5 References

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