



The Tame Automorphism And Jacobian Conjecture

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ABSTRACT

Consider the notations for this paper as: R_x indicates a commutative ring, though in most result R_x are going to be domain, and $R_x[X] := R_x[X_1, \dots, X_n]$ the Polynomial ring in n elements over R_x .

Here assuming that subsequent subcategories of $Aut_{R_x} R_x[X] := Aff(R_x, n)$ is equal the affine subgroup of $Aut_{R_x} R_x[X]$ including of every R_x -automorphisms F so that $deg F_i = 1 \forall i$. $J(R_x, n) :=$ the “de Jonquière’s” subgroup of $Aut_{R_x} R_x[X]$ including the R_x -automorphisms F Of the arrangement.

$F = (a_1X_1 + f_1(X_2, \dots, X_n), a_2X_2 + f_2(X_3, \dots, X_n), \dots, a_nX_n + f_n)$ Where one of the $a_i \in R_x^*$ and $f_i \in R_x[X_{i+1}, \dots, X_n]$ for every $1 \leq i \leq n-1$ and $f_n \in R_x$. $E(R_x, n) :=$ the subgroup of $Aut_{R_x} R_x[X]$ The Elementary Automorphism generated, that is the form of the automorphisms is $F = (X_1, \dots, X_{i-1}X_i + a, X_{i+1}, \dots, X_n)$ for some $a \in R_x[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$ and $1 \leq i \leq n$.

$T(R, n) :=$ the tame subset of $Aut_{R_x} R_x[X]$ The subgroup generated by is $Aff(R_x, n)$ and $E(R_x, n)$.

We get each part of $J(R_x, n)$ is a multiplication of a part of (R_x, n) and Elementary Automorphism finite in numbers. Therefore $J(R_x, n) \subset T(R_x, n)$. Also, coupling the “de Jonquiere’s” automorphisms with appropriate one effectively verifies permutation maps that all Elementary Automorphism fit in to the subgroup of $Aut_{R_x} R_x[X]$ created by $Aff(R_x, n)$ and $J(R_x, n)$. Hence, having $T(R_x, n) = (Aff(R_x, n) \text{ and } J(R_x, n))$.

In this paper here we assume the condition $n = 2$ and consider the R_x a domain. In this paper proving that the $T(R_x, 2)$ is the free merged result of $Aff(R_x, 2)$ and $J(R_x, 2)$ via their intersection. Moreover, we define an algorithm that determines if there is an endomorphism of polynomial of $R_x[X, Y]$ is tame.

By means of this process, the paper demonstrate that if R_x it is not a field, so it $T(R, 2) \neq Aut_{R_x} R_x[X, Y]$. But, consider R_x may be a field then it seems that we have impartiality, that is each in dimension two, automorphism taken over a field is tame. This is the more popular “Jung-van der Kulk theorem” (1.1.11).

Keywords : Automorphisms , Endomorphism of polynomial , Jacobian Conjecture.

Introduction:

In science, the Jacobian Conjecture is a well-known unsolved topic on polynomials in a several variables. It expresses that if a polynomial mapping from an n -dimensional field to itself has Jacobian Condition (i.e. $\det JF \in k^*$), then the mapping has a polynomial inverse. It was first guessed in 1939 by “Ott-Heinrich Keller”, and broadly worked by “Shreeram Abhyankar”, as a challenging problem in Algebraic geometry that can be solved using some basic knowledge of calculus.

The Jacobian Conjecture is infamous for the huge number of endeavoured confirmations that ended up containing unpretentious mistakes. Starting at 2018, there are no conceivable professes to have demonstrated it. Indeed, even the two-variable case has not solved. There are no known convincing explanations behind trusting it to be valid, and as per “Van Dan Essen” (1997) there are a few doubts that the guess is false for enormous quantities of factors. “The Jacobian Conjecture was numbered 16 in Stephen Smale’s 1998 rundown of Mathematical Problems for the Next Century”.

If $F: k^n \rightarrow k^n$ is a polynomial map and $\det JF \in k^*$. This map is called “Keller Map”

Then the Jacobian Conjecture is as follows:

Jacobian Conjecture: If we consider k as a field of characteristic zero.

If $F: k^n \rightarrow k^n$ is a “Keller Map”, then F is invertible.

If as indicated by “Van Dan Essen” (1997), the issue was first guessed by “Keller” in 1939 for the restricted case of two variables and integer coefficients.

Visibly simple the problem of Jacobian Conjecture comes up short if k has characteristic $p > 0$, even for the case of one variable. The field characteristic should be prime, so it must be ≥ 2 . The polynomial $x - x^p$ has differentiation $1 - px^{p-1}$ Which is 1 (since px is 0) but inverse of such a simple function does not exists. “Adjmagbo” (1995) recommended that one can extend Jacobian Conjecture to characteristic $p > 0$ by including the theory that p doesn’t isolate the level of the field extension $k(X)/k(F)$.

The condition $JF \neq 0$ is identified with the converse capacity hypothesis in multivariable calculus. Truth be told for smooth mapping functions (thus specifically for polynomials) a smooth neighbourhood inverse of F exists at each point where JF is non-zero. For instance, the function $F(x) = x + x^3$ is having inverse, however the inverse isn’t polynomial.

1. 1 Integral Domain of Two Variables Tame Automorphism Group.

Entire of this part R_x indicates a domain that is integral. Let us Firstly, explain that $T(R_x, 2)$. The free amalgamated item is if $Aff(R_x, 2)$ and $J(R_x, 2)$ on their juncture. Moreover, describing a process that decides if a given endomorphism F of

$R_x[X, Y]$ is tame. The same process we able to use to decompose F as an effect of elementary and linear automorphisms, in case F is tame. We deduce that in case R is not a field, then $T(R_x, 2)$ is a proper subgroup of $Aut_{R_x} R_x[X, Y]$. On the other hand, if R_x is a field, we will show that

$Aut_{R_x} R_x[X, Y] = T(R_x, 2)$ and hence we get $Aut_{R_x} R_x[X, Y]$ as a amalgamated free expression over $Aff(R_x, 2)$ and $J(R_x, 2)$ intersection Considering the theorem of “Jung-van der Kulk”, moreover identified as the theory of “Automorphism”. We need the following lemmas to prove the outcomes mentioned above.

Lemma 1. 1. 1: Considering G as a group of dual subgroups formed by H and K . Next all components g of G can be written as $g = h_0 k_1 h_1 \dots k_\ell h_\ell$
For certain $\ell \geq 1$, where $h_1 \in H/K$ for all $1 \leq i \leq \ell - 1$ and $k_1 \in K/H \forall 1 \leq i \leq \ell$ and $h_0 \in H$.

Proof: Consider $g \in G$ we are having $h_0 k_1 h_1 \dots k_\ell h_\ell$ for some $\ell \geq 1, h_i \in H$ and $k_i \in K$. If the extra provision on the h_i and k_i is not satisfied, then we can get an expression of the same sort for g but with ℓ substituted by $\ell - 1$ as follow: If, for instance, $h_{i_0} \in K$ for some $1 \leq i_0 \leq \ell - 1$ then

$g = h_0 k_1 \dots h_{i_0-1} (k_{i_0-1} h_{i_0} k_{i_0+1}) h_{i_0+1} \dots k_\ell h_\ell$
And $k_{i_0-1} h_{i_0} k_{i_0+1} \in K$. To get required expression for g we perform finite number of such reductions. For formulating 1.1.2 below, and with the algorithm released, we need the following notation:

Let $F = (F_1, F_2) \in R_x[R_x, Y]^2$. Then
 $bideg F := (deg F_1, deg F_2)$, $tdeg F := deg F_1 + deg F_2$.

Now let $F \in T(R_x, 2)$. Then applying 1.1.1 to

$$G := T(R_x, 2), H := Aff(R_x, 2),$$

$K := J(R_x, 2)$ and $g := F$ we can write

$$F = \lambda_0 \tau_1 \lambda_1 \dots \tau_\ell \lambda_\ell, \text{ With } \lambda_i \in Aff(R, 2)/J(R, 2) \text{ for all } 1 \leq i \leq \ell - 1 \text{ and } \tau_i \in J(R_x, 2)/Aff(R_x, 2), \forall 1 \leq i \leq \ell.$$

Write $\lambda_i = (a_i X + b_i Y + c_i, d_i X + e_i Y + f_i)$.

Lemma 1. 1. 2: $\forall 1 \leq i \leq \ell$ We have $bide(\tau_i \lambda_i \dots \tau_\ell \lambda_\ell) = (\prod_{j=i}^\ell deg \tau_j, \prod_{j=i+1}^\ell deg \tau_j)$.

The second product is of F if $i = \ell$.

Proof: Via reducing theory of induction upon i . the condition $i = \ell$ is observable. Let us presume, then, that the declaration is valid for certain $2 \leq i \leq \ell$ and assume $bidge(\tau_{i-1} \lambda_{i-1} \dots \tau_\ell \lambda_\ell)$.

First observe that, since $\lambda_{i-1} \notin J(R_x, 2)$ we have that $d_{i-1} \neq 0$. Consequently

$bideg(\lambda_{i-1} \tau_i \lambda_i \dots \tau_\ell \lambda_\ell) = (p_i, \prod_{j=i}^\ell deg \tau_j)$ where

$p_i \leq \prod_{j=i}^\ell deg \tau_j$. Thence, since $\tau_{i-1} \notin Aff(R_x, 2)$ we have that $deg \tau_{i-1} \geq 2$. So $bidge(\tau_{i-1} \lambda_{i-1} \tau_i \lambda_i \dots \tau_\ell \lambda_\ell) = (\prod_{j=i}^\ell deg \tau_j, \prod_{j=i}^\ell deg \tau_j)$ which concludes the Proof.

Corollary 1. 1. 3:

Consider $T(R_x, 2)$ as a combined unrestricted product of $Aff(R_x, 2)$ and $J(R, 2)$.

Above their juncture, i.e. $T(R_x, 2)$ these two groups are generated and if $\tau_j \in J(R_x, 2) \setminus Aff(R_x, 2)$ and $\lambda_i \in Aff(R, 2) \setminus J(R, 2)$ then $\tau_1 \lambda_1 \dots \tau_n \lambda_n \tau_{n+1}$ does not belongs to $Aff(R, 2)$.

Proof: As noticed above, $T(R_x, 2)$ is created by $Aff(R_x, 2)$ and $J(R_x, 2)$.

Supposing

$\tau_1 \lambda_1 \dots \tau_n \lambda_n \tau_{n+1} = \lambda \in Aff(R_x, 2)$ with
 $\tau_j \in J(R_x, 2) \setminus Aff(R_x, 2)$ and $\lambda_i \in Aff(R, 2) \setminus J(R, 2)$ for all i .

Then $\tau_1 \lambda_1 \dots \tau_n \lambda_n \tau_{n+1} \lambda^{-1} = (X, Y)$.

So $bidge(\tau_1 \lambda_1 \dots \tau_{n+1} \lambda^{-1}) = (1, 1)$. otherwise it follows from 1.1.2 that $bidge(\tau_1 \lambda_1 \dots \tau_{n+1} \lambda^{-1}) = (\prod_{j=1}^n deg \tau_j, \prod_{j=2}^n deg \tau_j)$.

Thus, this bidegree is unequal to $(1, 1)$.

since $deg \tau_j \geq 2$, a contradiction. (1.1.4)

Note: Let J_0 Become a subgroup of $J(R_x, 2)$ the elements are composed of $(X + g(Y), Y)$ Along with $g(Y) \in R_x[Y]$. One easily see that $T(R_x, 2)$ is generated by $Aff(R_x, 2)$ & J_0 . Though $T(R_x, 2)$ It is not the free amalgamated result of $Aff(R_x, 2)$ and J_0 over their intersection: For instance, $\tau_1 = (X - Y^2, Y), \lambda_1 = (X, Y + 1)$ and $\tau_2 = (X + (Y + 1)^2, Y)$. Then $\tau_1 \lambda_1 \tau_2 = \lambda_1 \in Aff(R, 2)$ but none of $\tau_1, \lambda_1, \tau_2$ fit in to $Aff(R_x, 2)$ and J_0 intersection.

Remark 1. 1. 5:

Consider R isn't a domain $T(R_x, 2)$ It is not the free amalgamated result of $Aff(R_x, 2)$ and $J(R_x, 2)$ over their intersection.

Corollary 4.1.6: Let

$F_1 = (F_1, F_2) \in T(R_x, 2)$ with $bideg F = (d_1, d_2)$. Let h_1 Refer to the homogeneous components of F_1 $f \deg d_1$.

Then $d_1 | d_2$ or $d_2 | d_1$.

if $\deg F > 1$, then we have

If $d_1 < d_2$, then $h_2 = ch_1^{d_2/d_1}$, for some $c \in R_x$.

If $d_2 < d_1$, then $h_1 = ch_2^{d_1/d_2}$, for some $c \in R_x$.

If $d_1 = d_2$, then there exists $\lambda \in Aff(R, 2)$ such that

$\deg F'_1 > \deg F'_2$, where $(F'_1, F'_2) := \lambda \circ F$.

Proof: By 1.1.2 we have $F = \lambda_0 \tau_1 \lambda_1 \dots \tau_\ell \lambda_\ell$ with

$bidge(\tau_1 \lambda_1 \dots \tau_\ell \lambda_\ell) = (\prod_{j=1}^\ell \deg \tau_j, \prod_{j=2}^\ell \deg \tau_j)$. Which provides a) Furthermore b) go alone with considering 1), 2) or 3) according to

$a_0 = 0, d_0 = 0$ or $a_0 d_0 \neq 0$ where $\lambda_0 = (a_0 X + Y b_0 + c_0 d_0 X + e_0 Y + f_0)$.

Remark 1.1.7: Indication as in 1.1.6. If $d_1 = d_2$ Next overall, there is no $c \in R$ so that $h_1 = ch_2$ or $h_2 = ch_1$.

Remark 1.1.8: Consider R the argument is not a domain, a) of 4.1.6 is untrue. Monitor that consider F_1 is a 1) (resp. 2)) of 1.1.6, then

$tdeg \tau^{-1} \circ F_1 < tdeg F_1$, Where $\tau = (X, Y + cX^{d_1/d_2})$ (resp. $\tau = (X + Y^{d_1/d_2}, Y)$).

So, we are having: Process to Choose if $F = (F_1, F_2) \in R_x[X, Y]^2$ belongs to $T(R_x, 2)$.

Key: $F = (F_1, F_2) \in R_x[X, Y]^2$.

1. Let $(d_1, d_2) = bideg(F_1, F_2)$.

2. If $d_1 = d_2 = 1$, go to 7.

3. If $d_1 \neq d_2$, go to 5.

4. If there exists $\lambda \in Aff(R_x, 2)$ with

$tdeg \lambda F < tdeg F$, replace by λF and go to 1, else stop:

$F \notin T(R_x, 2)$.

5. If $d_1 < d_2$, replace F by (F_1, F_2) .

6. If $d_1 | d_2$ and there exists $c \in R_x$ with $ch_1^{d_1/d_2}$, replace F_1 by $(X, Y - cX^{d_1/d_2}) \circ F$ and to 1, else stop : $F \notin T(R_x, 2)$.

If $\det JF_1 \in R_x^*$, stop : $F \in T(R_x, 2)$, else stop : $F \notin T(R_x, 2)$.

7. If $\det JF \in R^*$, stop $F \in T(R, 2)$, else stop:

$F \notin (T, 2)$.

Now we will use this process, or 1.1.6 to tackle the following questions.

Question: According to which condition on R_x is every R_x – automorphism of $R_x[X, Y]$ tame, i.e. According to which condition on R_x do we have the equal to $T(R_x, 2) = Aut_{R_x} R_x[X, Y]$?

Proposition 1.1.9: If R_x is non field, then $T(R_x, 2) \neq Aut_{R_x} R_x[X, Y]$.

More precisely let $0 \neq z \in R_x$. Consider being a non-unit $F = (X - 2Y(zX + Y^2) - z(zX + Y^2)^2, Y + z(zX + Y^2))$. Then $F \in Aut_R R_x[X, Y] \setminus T(R_x, 2)$.

Proof: Let us consider $F = expD$, here D is the Locally nilpotent Derivation.

$$D = (zX + Y^2)(-2Y\partial_X + z\partial_Y) \text{ on } R_x[X, Y].$$

$$\text{So } F \in Aut_{R_x} R_x[X, Y].$$

If $F \in T(R_x, 2)$ then by 1.1.6 1) we get that $-zY^4 = c(zY^2)^2$ for some $c \in R_x$.

Therefore $-z = cz^2$, Or to divided by $-z$ we are having z is a unit in R_x , here we get a contradiction.

Remarks 1.1.10: Automorphism of the F_1 demonstrated in 1.1.9 is the known as “Nagata automorphism”, and introduction by “Nagata” in [78]. It is fascinating to remember that “Nagata” defied his compositional automorphism

$F = \sigma_1^{-1} \sigma_2 \sigma_1$, where $\sigma_1, \sigma_2 \in Aut_K K[X, Y]$

($K :=$ the quotient field of R_x) are defined by

$$\sigma_1(X, Y) := (X + z^{-1}Y^2, Y), \sigma_2(X, Y) := (X, Y + z^2X).$$

Now the noteworthy point is that the requisite condition is “ R_x is a field” It is also appropriate for freedom to have $Aut_{R_x} R_x[X, Y] = T(R_x, 2)$. That is to quote,

Theorem 1.1.11 (“Jung, van der Kulk”): If k_1 is a field, then

$Aut_{k_1} k_1[X, Y] = T(k_1, 2)$. More particularly, $Aut_{k_1} k_1[X, Y]$ is the amalgamated free multiplication of $Aff(k_1, 2)$ and $J(k_1, 2)$ Above their junction.

Considering the strong version of the “Rentschler's theorem” is the basis for the proof we can provide below. The considered Proof of 1.1.11 thus just fits for the case $\text{char } k_1 = 0$.

We apply to the articles for the general situation, [10] of “Van der Kulk”, [11]. Now onwards, k_1 denotes a field with zero characteristics.

Theorem 1. 1. 12(“Rentschler”):

Considering $0 \neq D1$ as a Locally nilpotent Derivation on $k1[X_1, X_2]$.

Afterward there exists

$$h \in T(k, 2) \text{ and } f(X_2) \in k1[X_2] \text{ s.t.}$$

$$h^{-1}D1h = f(X_2)\partial_{X_1}.$$

Let us explain how the "Jung-van der Kulk" theorem is inferred before we prove this outcome.

Proof Of 1. 1. 11: write $A := k1[X_1, X_2]$, 1.1.3 we only need to show that

$T(k1, 2) = \text{Aut}_{k1} A$. Therefore let $F = (F_1, F_2) \in \text{Aut}_{k1} A$. Then $\frac{\partial}{\partial F_1}$ is Locally nilpotent, so by 1.1.12 $\exists h \in$

$$T(k1, 2) \text{ \& } f(X_2) \in k1[X_2] \text{ s.t. } h^{-1} \frac{\partial}{\partial F_1} = f(X_2)\partial_{X_1}.$$

Considering $g \in A$,

$$\text{Then } h^{-1} \frac{\partial}{\partial F_1} h(g) = 0 \text{ if } h(g) = 0$$

$$\text{If } h(g) \in \ker \frac{\partial}{\partial F_1} = k[X_2], \text{ i.e. } \ker h^{-1} \frac{\partial}{\partial F_1} h = k[h^{-1}(F_2)] \text{ \& } \ker f(X_2)\partial_{X_1} = k1[X_2],$$

$$\text{So } k1[X_2] = k1[h^{-1}(F_2)] \Rightarrow h^{-1}(F_2) = cX_2 + d, \text{ for some } c \in k1^* \text{ and } d \in k.$$

$$\text{So } F_2 = ch(X_2) + d.$$

$$\text{Moreover } (h^{-1} \frac{\partial}{\partial F_1} h)(h^{-1}(F_1)) = 1,$$

$$f(X_2)\partial_{X_1}(h^{-1}(F_2)) = 1 \text{ that implies}$$

$$\partial_{X_1}(h^{-1}(F_2)) \in k^*, \text{ so } h^{-1}(F_2)c'X_2 + d'(X_2)$$

$$\text{for some } c' \in k1^* \text{ and } d'X_2 \in k1[X_2].$$

$$\text{Consequently } F_2 = c'h(X_2) + d'(h(X_2)).$$

Summarizing

$$(F_1, F_2) = (c'h(X_2) + d'(h(X_2)), ch(X_2) + d).$$

Since $h \in T(R_x, 2)$ we can prove that $F1 \in T(R_x, 2)$ which finishes the Proof.

Before we prove “Rentschler’s theorem”, giving certain simplifications regarding “Derand gradings”. The following consequence is the beginning.

Proposition 1. 1. 13:

Considering $R_x = \bigoplus_{m \in \mathbb{Z}} R_{x_m}$ as a ring with graded and $D1$ a non-zero Derivation on R_x .

Assuming that $D1$ the following form can be written As a finite amount of derivation of

$$D1 = D1_p + D_{p+1} + \dots + D1_d$$

so that $D1_n R_{x_m} \subset R_{x_{n+m}}$, for every $n, m \in \mathbb{Z}$. Consider $D1$ is Locally nilpotent so is $D1_d$.

Proof: It’s enough to prove that for each $m \in \mathbb{Z}$ every component of R_{x_m} is described by some power of D_d . So let $g \in R_{x_m}$. Hence $D1$ is Locally nilpotent, There is N in such a way that there is $D1^N g = 0$. The constituent of $D1^N g$ which $\in R_{x_{m+Nd}} = D1_d^N g$. Hence $D1^N g = 0$ it implies that $D1_d^N g = 0$, as required.

From now onwards we limit to the case

$$R_x := k1[X] = k1[X_1, \dots, X_2].$$

For Any single non-zero vector w in \mathbb{Z}^n By defining for each, we combine the w -grading on R $d \in \mathbb{Z} R_{x_d}(w)$ similarly $k1$ -vector R area created by all monomials X^a along with $\langle a, w \rangle = d$. Here $\langle \rangle$ indicated the standard inner multiplication on R_x^n . To avoid the confusion, we can use R_{x_d} for the $R_{x_d}(w)$.

For applying above 1.1.13 proposition we show

Proposition 1. 1. 14:

Considering $0 \neq w \in \mathbb{Z}^n$ and assume on R the w -grading. Considering $D1 \neq 0$ Der on R_x . Afterward we can write $D1$ as a finite addition of derivation

$$\sum D1_p \text{ such that } D1_d R_{x_d} \subset R_{x_{p+d}} \text{ for all } p, d \in \mathbb{Z}.$$

Proof:

The exclusivity follows from the definition of Derivation, therefore we only must show the presence.

So, let $T := cX^a \partial_i$ be a term appearing in $D1$. substitute $s := a - e_i$, where e_i denotes the i -th standard basis vector of \mathbb{R}^n .

Then $T(X^m) \in k1 X^{m+s} \forall m$. Calling s the strength of T and substitute ea finite decomposition $\text{supp } D1 = \{s \in \mathbb{Z}^n \mid D1 \text{ contain a term of strength } s\}$.

Take $D1(s)$ the sum of all the above set.

Therefore, we get

$$D1_p := \sum_{\langle s, w \rangle = p} D1(s).$$

Obviously $D1 = \sum D1_p$, since every non-zero term seeming in $D1$ had certain asset, known as s_0 and therefore $\in D1_{p_0}$ here $p_0 = \langle s_0, w \rangle$. So it remains to show that $D1_p R_{x_d} \subset R_{x_{p+d}}$ for every p, d or equal to $D1(s) X^m \in R_{x_{p+d}}$ for all $X^m \in$

R_{x_d} and all $s \in \mathbb{Z}^n$ with $\langle s, w \rangle = p$. To see this first, observe that $D1(s) X^m \in k1 X^{m+s}$, we demonstrate the $\langle m + s, w \rangle = p + d$.

But since then, this conveniently follows $\langle m + s, w \rangle = \langle m, w \rangle + \langle s, w \rangle = d + p$ using that $X^m \in R_{x_p}$.

The decomposition of $D1$ given in 1.1.13 is called w -homogeneous decomposition of $D1$. Consider p is max with $D1_p \neq 0$, p is known as w -deg of $D1$, denoted $wdegD1$. In case $w = (1, 1, \dots, 1)p$ is known as deg of $D1$ and indicated by $degD1$. The following proposition plays an important part in getting proof of ‘‘Theorem of Rentschler’s’’.

Proposition 1. 1. 15: Considering $R_x = \bigoplus_{d \in \mathbb{Z}} R_x$ any grading on it will be $R_x = k1[X]$, therefore It doesn't even need to be a w -grade, and Considering $D1 = \sum D1_p$ not an infinite decomposition sustaining $D1_p R_{x_d} \subset R_{x_{p+d}} \forall p, d \in \mathbb{Z}$.

Assuming p be greatest with $D1_p \neq 0$.

If $D1_p = D1(s)$ for certain $s = (s_1, \dots, s_n) \in \mathbb{Z}^n$, where all $s_i \geq 0$, then $D1$ is not LN.

Proof : Considering $\in \mathbb{Z}^n$. We from following remark that $D1(s) = X^s \sum c_j X_j \partial_j$ for some $c_j \in k$. Since $D1(s) = D1_p \neq 0$ J exists with the $c_j \neq 0$. Observing that if all $s_i \geq 0$, then $D(s)(X_j) = c_j X^s X_j$ implies that $c_j X^s$ is a positive eigenvalue of

$D1(s)$ in $k1[X]$. Thus, from 1.8.32 $D1(s)$ is not Locally nilpotent, i.e. $D1_p$ is not Locally nilpotent Before applying 1.1.13 Consider the condition $n = 2$ and write $R_x = k1[X, Y]$ in its place of $k1[X_1, X_2]$. Considering $D1 \neq 0$ be a Der on R_x . Then denoting an element of $\text{supp } D1$ as (s, t) . Thus $s, t \geq -1$ and e.g. $(s, -1) \in \text{supp } D1$ That implies that $D1$ includes a term of the form $cX^s \partial_j$ with $c \in k^*$.

Corollary 1. 1. 16:

Consider $D1$ is Locally nilpotent, then either

$D1 = f'(Y) \partial_x$, for some $f'(Y) \in k1[Y]$ or

$D1 = f'(X) \partial_y$, for some $f'(X) \in k1[X]$ or

there exists $s_0, t_0 \geq 0$ such that $(s_0, -1)$ and $(-1, t_0)$ belong to $\text{supp } D1$ and furthermore $\text{supp } D1$ is connected in the triangle with vertices $(s_0, -1), (-1, -1), (-1, t_0)$.

Proof : Let $p_0 := \text{deg } D1$. So $p_0 \geq -1$. If $p_0 = -1$ then $D1 = a\partial_x + b\partial_y$ for few

$a, b \in k1$, both not zero and we're completed with the proof. Therefore, now considering $p_0 \geq 0$ and Considering ℓ indicated the line $x + y = p_0$

Case I: ℓ includes a point of the method $(-1, t_0)$. By description of ℓ all the points of a point $\text{supp } D1$ is on or under this axis. Turn ℓ around the dot now $(-1, t_0)$ (clockwise) until one hit another point of $\text{supp } D1$. Consider ℓ about more than one supply point $D1$ We don't have to turn ℓ , i.e. we change it over an angle of degree zero. In case one does not meet any other point of $\text{supp } D1$, afterward clearly

$\text{supp } D1 = \{(-1, t_0)\}$ and consequently $D1 = cY^{t_0} \partial_x$ for few $c \in k1^*$ and we are in case 1). If one only meets points of the form $(-1, t)$ then all these points satisfy $t \leq t_0$. Hence Even, we are in the condition 1). We should then conclude that we have at least one segment $(s, t) \in \text{supp } D1$ with $s \geq 0$. Given We have these points to select second coordinate minimal. Label this point (s', t') . We demonstrated that $t' = -1$, what substitute in case 3). Hence assume that $t' \geq 0$. we will come across a contradiction. We know that $s' \geq 0$, so together $s', t' \geq 0$. Now move ℓ over a tiny angle around (s', t') (clockwise) we notice that a line appears, $\ell' : \omega_1 x + \omega_2 y = d$ with $\omega_1, \omega_2, d \in \mathbb{Z}$ such that (s', t') is the solitary point of $\text{supp } D1$ On this line, and in addition, all other points of $\text{supp } D1$ are under ℓ' . substituting $\omega := (\omega_1, \omega_2)$ and $p := \omega \text{deg } D1$, it follow that $D1_p = D1((s', t'))$. Applying 1.1.15 demonstrate that $D1$ is not Locally nilpotent, a contradiction. Hence $t' = -1$.

CASE II: ℓ Include a point in the form of $(s', -1)$. This condition is same as case I.

CASE III: ℓ Include either a point in the form of $(s, -1)$ or a point in the form of

$(-1, t)$. Considering $(s, t) \in \ell$ with t min. Hence $s, t \geq 0$ We will see a line once again, ℓ' so that (s, t) the single point of the $\text{supp } D1$ on ℓ' and entirely other points of $\text{supp } D1$ are under ℓ' . Next, as over, we are having a negation, that summarises the proof.

Proof Of ‘‘Rentscher’s Theorem’’

1) Let's use 1.1.16 & consider $D1 = f(X) \partial_y$ take $h = (Y, X)$.

Afterward $h^{-1} D1 h = f(Y) \partial_x$. Therefore we consider that we are having s_0, t_0 in 4.1.16. We can write $s_0(D1), t_0(D1)$ in its place of s_0 , correspondingly t_0 if essential.

Considering ℓ' be the line crossing through both $(s_0, -1)$ and $(-1, t_0)$. Hence ℓ' is provided by the eq $(t_0 + 1)x + (s_0 + 1)y = p$, where $p = s_0 t_0 - 1$.

Considering $\omega := (t_0 + 1, s_0 + 1)$.

Therefore by $\omega \text{deg } D = p$ and by 1.1.13

$D1_p$ is Locally nilpotent. Inscribe

$D1_p = gD1_1$ where $D1_1 = a\partial_x + b\partial_y$

fulfils $\text{gcd}(a, b) = 1$. By $D1_1$ is Locally nilpotent and $D1_1(g) = 0$. since $D1_p$ is w -homogenous.

2) Since $D1_1$ is Locally Nilpotent with

$gcd(a, b) = 1$ it tracks from that $D1_1$ has a piece of it in the $k1[X, Y]$,
Thus in specific $a(0) \neq 0$ or $b(0) \neq 0$.

We may assume that

$a(0) \neq 0$. So $D1_1$ contain a term of the form $c\partial_X$ with $c \in k^*$.

Since $(s_0, -1) \in \text{supp } D1_p$, i.e. $D1_p$ contain a . The concept of the form $c'X^{s_0}\partial_X$ and since $D1_p = gD1_1$ it follows that $D1_1$ also contains a concept in the form $dX^r\partial_Y$ with $r \geq 0$ and $d \in k1^*$. But then follows that $D1_1 = c\partial_X + dX^r\partial_Y$, namely $D1_1$

w – homogenous whence $\text{supp } D1_1$ It's on a line that is, Obviously, the line that goes through $(-1,0)$ and $(r, -1)$. This line, however, does not contain any other integer points.
co-ordinates ≥ -1 .

3) since $D1_1(g) = 0$ we get that $g \in \text{ker}$

$$D1_1 = k\left[Y - \frac{d}{(r+1)c} X^{r+1}\right].$$

The homogeneity of g implies that

$$g = a\left(Y - \frac{d}{(r+1)c} X^{r+1}\right)^N \text{ for some}$$

$a \in k^*$ and $N \in \mathbb{N}$.

$$\text{So } D1_p = a\left(Y - \frac{d}{(r+1)c} X^{r+1}\right)^N (c\partial_X + dX^r\partial_Y).$$

4) Lastly Considering h be the automorphism provided by $h(X) = X$
and $h(Y) = Y - \frac{d}{(r+1)c} X^{r+1}$.

Then one easily verifies that $h^{-1}D1_ph = aY^N c\partial_X$. so single effortlessly demonstrated that

$h(R_{x_d}) \subset R_{x_d}$ and therefore $h(R_{x_d}) = R_{x_d}$ for each $d \in \mathbb{Z}$. Accordingly

$(h^{-1}D1h)_m = h^{-1}D1_m h \forall m \in \mathbb{Z}$. Particularly the w – degree of $h^{-1}D1h$ is equality to

ω – degree of $D1$ which equal to p . Therefore $h^{-1}D1_ph = aY^N c\partial_X$ For a point in the form, there is no contribution $(s, -1)$ in $\text{supp } h^{-1}D1h$ impending from $(h^{-1}D1h)_p$. Consequently $s_0(h^{-1}D1h) < s_0(D1)$. The theorem then proceeds with induction on $s_0(D1) + t_0(D1)$.

Remark 1.1.17: Further proof of equality $\text{Aut}_{k1}k1[X, Y] = T(k1,2)$ is given in the next section of this section, see 1.3.6. This proof is a result of the ‘‘Abhyankar-Mohs’’ theorem.

As result of we get some information about a co-ordinate in the above proof $k1[X, Y]$.

First if $0 \neq f = \sum f_{ij}X^iY^j \in k1[X, Y]$ we put $\text{supp } f := \{(i, j) \in \mathbb{Z}^2 \mid f_{ij} \neq 0\}$.

The ‘‘Newton polygon’’ of f , represented as $N(f)$, is the convex hull of $\text{supp } f \cup \{0,0\}$. Lastly saying that f has $r \geq 1$ points on infinity if f_+ , the highest deg homogeneous part of f giving to the $(1,1)$ – grading has r distinct (linear) prime factors in $k1[X, Y]$.

Proposition 1.1.18: If f is a co-ordinate in $k1[X, Y]$ then

1) $N(f)$ is a triangle with vertices $(0,0)$, $(n, 0)$, $(0, m)$, where n, m are integers ≥ 0 .

2) f has one point at infinity.

Proof: 1) Since f is a coordinate there exists $g \in k1[X, Y]$ s.t $k1[X, Y] = k1[f, g]$ and $\det(f, g) = 1$.

Hence $D1 := \frac{\partial}{\partial g}$ is Locally nilpotent on $k1[X, Y]$ and

$D1 = f_Y\partial_X - f_X\partial_Y$. Using this description of $D1$ we get that

$(i, j) \in \text{supp } f$ if $(i - 1, j - 1) \in \text{supp } D1$.

Using 1.1.16 We discover that a triangle is $N(f)$. 2) By 1) We realise that a triangle is $N(f)$.

consider $n > m$ then $f_+ = c_1X^n$ for some $c_1 \in k1^*$ and if $n < m$ clearly

$f_+ = c_2Y^m$ with $c_2 \in k^*$. Lastly, if $n = m$, afterward giving to the $(1,1)$ -grading we get $D1_p = f_{+Y}\partial_X - f_{+X}\partial_Y$, here p is the $(1,1)$ -deg of $D1$.

Since from the proof of above Theorem, we see that

$$D1_p = a\left(Y - \frac{d}{(r+1)c} X^{r+1}\right)^N (c\partial_X + dX^r\partial_Y).$$

for some $a, c, d \in k^*$, $N \geq 0$ and $r \geq 0$.

Also $Y - \frac{d}{(r+1)c} X^{r+1}$ is $(1,1)$ -homogeneous.

So, $r = 0$ and consequently and

$$D1_p = a\left(Y - \frac{d}{c} X\right)^N (c\partial_X + d\partial_Y).$$

Since, as observed above, $D1_p = f_{+Y}\partial_X - f_{+X}\partial_Y$ we obtain

$$f_+ = ac \frac{1}{N+1} \left(Y - \frac{d}{c} X\right)^{N+1}. \text{ Hence the Proof.}$$

Remarks: 1) Let R'_x be a CR that is non integral domain is, considering $ab = 0$ here a and b these are positive components in R'_x .

a) Letting $\tau := (X + aY^2, Y)$ and

$\lambda := (X, Y + bX)$. Then both τ and λ do not belong to $Aff(R'_x, 2) \cap J(R_x, 2)$ and that $\tau \lambda \tau^{-1} = \lambda$.

So $T(R'_x, 2)$ is not the free outcome of $Aff(R'_x, 2)$ and $J(R'_x, 2)$ on their intersection.

b) Let $F = (X, Y + aX^2) \circ (X + aY^2, Y)$.

Then

$F = (X + aY^2, Y + aX^2)$, So $(d_1, d_2) := \text{bideg } F = (2, 3)$ and d_1 does not divide d_2 .

2) suppose that $R1_x$ is a domain which is not a field and $z \in R1_x$ is such that both z and $1 - z$ are non-units in $R1_x$.

Put $F = \tau \lambda$, where $\lambda = ((1 - z)X + zY, -zX + (1 - z)Y)$ and $\tau = (X + Y^2, Y)$.

a) Then $\lambda \in \text{Aff}(R1_x, 2)$,

$F \in T(R1_x, 2)$ and $\text{bideg } F = (2, 2)$.

b) Let $h_1 h_2$ be as in 1.1.6. Then there does not exist $c \in R1_x$ such that $h_1 = ch_2$ or $h_2 = ch_1$.

3) Letting z a component in R_x and $F1$ the "Nagata automorphism" defined in 1.1.9. Then that $F \in T(R_x, 2)$.

4) Let R_x be a ring encompassing the element a, b , Assuming $f(T) \in R_x[T]$ and define

$F := (X + bf(aX + bY), Y - af(aX + bY))$. a) Then $F1 \in \text{Aut}_{R_x} R_x[X, Y]$. [we get this from $D1 := f(aX + bY)(b\partial_X - a\partial_Y)$

is a Locally nilpotent Derivation on $R_x[X, Y]$ and that $F = \exp D1$.] b) Now assume that R_x is a domain such that $R_x a + R_x b$ is not a principal ideal. Consider $\text{deg } f(T) \geq 2$, then $F1 \notin T(R_x, 2)$.

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