The Tame Automorphism And Jacobian Conjucture

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ABSTRACT

Consider the notations for this paper as: R_x indicates a commutative ring, though in most result R_x are going to be domain, and R_x [X]:= $R_x[X_1,\dots,X_n]$ the Polynomial ring in n elements overR_x.

Here assuming that subsequent subcategories of $Aut_{R_x}R_x[X]$::$Aff(R_x$,n) is equal the affine subgroup of $Aut_{R_x}R_x[X]$ including of every R_x-automorphisms F so that $deg\ F_i = 1\ \forall\ i.\ J\ (R_x,n) :=$ the "de Jonquière's" subgroup of $Aut_R\ R_x$ $[X]$ including the R_x-automorphisms F Of the arrangement.

$F = (a_1X_1 + f_1(X_2,\dots,X_n)a_2X_2 + f_2(X_3,\dots,X_n)\ ,\dots,\ a_nX_n + f_n)$ Where one of the $a_i \in R^*$ $and f_i\ \in R_x\ [X_{i+1},\dots,X_n]$ for every $1\ \le i \le n-1\ and\ f_n \in R_x.E(R_x,n) :=$ the subgroup of $Aut_{R_x}\ R_x\ [X]$ The Elementary Automorphism generated, that is the form of the automorphisms is $F=\ (X_1,\dots,X_{i-1}X_i + a\ ,X_{i+1},\dots,X_n,)$ for some $a\ \in R_x[X_1,\dots,\widehat{X_\iota},\dots,X_n,]$ and $1 \le i \le\ n.$

$T(R\ ,n) :=$ the tame subset of $Aut_{R_x}R_x[X]$The subgroup generated by is $Aff(R_x$, n) and $E(R_x,n)$.

We get each part of $J\ (R_x$, n) is a multiplication of a part of$(R_x,\ n)$ and Elementary Automorphism finite in numbers. Therefore $J(R_x$, n) ⊂ $T\ (R_x$, n). Also, coupling the "de Jonquiere's" automorphisms with appropriate one effectively verifies permutation maps that all Elementary Automorphism fit in to the subgroup of $Aut_{R_x}\ R_x\ [X]$ created by $Aff\ (R_x$, n) and J (R_x, n). Hence, having $T\ (R_x$,n) =($Aff(R_x$,n) and J (R_x ,n)).

In this paper here we assume the condition $n\ =\ 2$ and consider the R_x a domain. In this paper proving that the $T\ (R_x$, 2) is the free merged result of $Aff\ (R_x$, 2) and $J\ (R_x$, 2) via their intersection. Moreover, we define an algorithm that determines if there is an endomorphism of polynomial of $R_x\ [X,Y]$ is tame.

By means of this process, the paper demonstrate that if R_x it is not a field, so it $T(R,2) \neq Aut_{R_x}\ R_x\ [X,Y]$. But, consider R_x may be a field then it seems that we have impartiality, that is each in dimension two, automorphism taken over a field is tame. This is the more popular "Jung-van der Kulk theorem" (1.1.11).

Keywords : Automorphisms , Endomorphism of polynomial , Jacobian Conjecture.

Introduction:

In science, the Jacobian Conjecture is a well-known unsolved topic on polynomials in a several variables. It expresses that if a polynomial mapping from an n-dimensional field to itself has Jacobian Condition (i.e. det$JF \in k^*$), then the mapping has a polynomial inverse. It was first guessed in 1939 by "Ott-Heinrich Keller", and broadly worked by "Shreeram Abhyankar", as a challenging problem in Algebraic geometry that can be solved using some basic knowledge of calculus.

The Jacobian Conjecture is infamous for the huge number of endeavoured confirmations that ended up containing unpretentious mistakes. Starting at 2018, there are no conceivable professes to have demonstrated it. Indeed, even the two-variable case has not solved. There are no known convincing explanations behind trusting it to be valid, and as per "Van Dan Essen" (1997) there are a few doubts that the guess is false for enormous quantities of factors. "The Jacobian Conjecture was numbered 16 in Stephen Smale's 1998 rundown of Mathematical Problems for the Next Century".

If $F: k^n \to k^n$ is a polynomial map and det $JF \in k^*$. This map is called "Keller Map"

Then the Jacobian Conjecture is as follows:

Jacobian Conjecture: If we consider k as a field of characteristic zero.

If $F: k^n \to k^n$ is a "Keller Map", then F is invertible.

If as indicated by "Van Dan Essen" (1997), the issue was first guessed by "Keller" in 1939 for the restricted case of two variables and integer coefficients.

Visibly simple the problem of Jacobian Conjecture comes up short if k has characteristic p > 0, even for the case of one variable. The field characteristic should be prime, so it must be ≥ 2. The polynomial $x - x^p$ has differentiation $1 - px^{p-1}$ Which is 1 (since px is 0) but inverse of such a simple function does not exists. "Adjamagbo" (1995) recommended that one can extend Jacobian Conjecture to characteristic p $>$ 0 by including the theory that p doesn't isolate the level of the field extension $k(X)/k(F)$.

The condition JF ≠ 0 is identified with the converse capacity hypothesis in multivariable calculus. Truth be told for smooth mapping functions (thus specifically for polynomials) a smooth neighbourhood inverse of F exists at each point where JF is non-zero. For instance, the function $F(x) =$ x +x^3 is having inverse, however the inverse isn't polynomial.

1. 1 Integral Domain of Two Variables Tame Automorphism Group.

Entire of this part R_x indicates a domain that is integral. Let us Firstly, explain that $T(R_x, 2)$. The free amalgamated item is if $Aff(R_x,2)$ and J $(R_x,2)$ on their juncture. Moreover, describing a process that decides if a given endomorphism F of

R_x $[X, Y]$ is tame. The same process we able to use to decompose F as an effect of elementary and linear automorphisms, in case F is tame. We deduce that in case R is not a field, then T $(R_x,2)$ is a proper subgroup of Aut_{R_x} R_x [X,Y]. On the other hand, if R_x is a field, we will show that

Aut_{R_x} R_x [X, Y] = T $(R_x,2)$ and hence we get Aut_{R_x} R_x [X,Y] as a amalgamated free expression over $Aff(R_x,2)$ and J $(R_x$, n) intersection Considering the theorem of "Jung-van der Kulk", moreover identified as the theory of "Automorphism". We need the following lemmas to prove the outcomes mentioned above.

***Lemma* 1. 1. 1:** Considering G as a group of dual subgroups formed by H *and* K. Next all components g of G can be written as g = $h_0 k_1 h_1 \ldots k_\ell\ h_\ell$

For certain $\ell \geq 1$, *where* $h_1 \in H/K$ for all $1 \leq i \leq \ell - 1$ *and* $k_1 \in K/H$ $\forall 1 \leq i \leq \ell$ and $h_0 \in H$.

Proof: Consider $g \in G$ we are having $h_0 k_1 h_1 \ldots k_\ell$ *f*or some $\ell \geq 1, h_i \in H$ and

$k_i \in K$. If the extra provision on the h_i *and* k_i is not satisfied, then we can get an expression of the same sort for g but with ℓ substituted by $\ell - 1$ as follow: If, for instance, $h_{i0} \in K$ for some $1 \leq i_0 \leq \ell - 1$ *then*

$g = h_0 k_1 \ldots h_{i0-1}(k_{i0-1} h_{i0} k_{i0+1}) h_{i0+1} \ldots k_\ell h_\ell$

And $k_{i0-1} h_{i0} k_{i0+1} \in K$. To get required expression for g we perform finite number of such reductions. For formulating 1.1.2 below, and with the algorithm released, we need the following notation:

Let $F = (F_1, F_2) \in R_x\ [R_x, Y]^2$. Then

$bidegF1 := (degF_1, degF_2),\ tdegF := degF_1 + degF_2.$

Now let $F \in T(R_x, 2)$. Then apllying 1.1.1 to

$$G := T(R_x, 2), H := Aff\ (R_x, 2),$$

$K := J(R_x, 2)$ and $g := F$ we can write

$F = \lambda_0 \tau_1 \lambda_1 \ldots \tau_\ell \lambda_\ell$, With $\lambda_i \in Aff\ (R, 2)/J$(R, 2) for all $1 \leq i \leq \ell - 1$ and $\tau_i \in J(R_x, 2)/$Aff$(R_x,2)$,$\forall 1 \leq i \leq \ell$.

Write $\lambda_i = (a_i$ X + $b_i Y + c_i, d_i$X + e_i Y + f_i).

***Lemma* 1. 1. 2:** $\forall\ 1 \leq i \leq \ell$ We have $bide(\tau_i \lambda_{\text{i}} \ldots \tau_\ell \lambda_\ell)$ = $\left(\prod_{j=i}^{\ell} deg\ \tau_j, \prod_{j=i+1}^{\ell} deg\ \tau_j\right)$.

The second product is of F if $i = \ell$.

Proof: Via reducing theory of induction upon i. the condition $i = \ell$ is observable. Let us presume, then, that the declaration is valid for certain $2 \leq i \leq \ell$ and assume $bidge\ (\tau_{i-1} \lambda_{i-1} \ldots \tau_\ell \lambda_\ell)$.

First observe that, since $\lambda_{i-1} \notin J\ (R_x, 2)$ we have that $d_{i-1} \neq 0$. Consequently

$bideg\ (\ \lambda_{\text{i}-1} \tau_i \lambda_i .. \tau_\ell \lambda_\ell) = (p_i, \prod_{j-i}^{\ell} \deg \tau_j)$ where

$p_i \leq \prod_{j-i}^{\ell} \deg \tau_j$ .Thence, since $\tau_{i-1} \notin Aff(R_x, 2)$ we have that deg $\tau_{i-1} \geq 2$. So $bidge(\tau_{i-1} \lambda_{i-1} \tau_i \lambda_i \ldots \tau_\ell \lambda_\ell) = \prod_{j=i}^{\ell} \deg \tau_j, \prod_{j-i}^{\ell} \deg \tau_j)$ which concludes the Proof.

***Corollary* 1. 1. 3:**

Consider $T(R_x, 2)$ as a combined unrestricted product of $Aff\ (R_x, 2)$ and $J\ (R, 2)$.

Above their junction, i.e. $T\ (R_x, 2)$ these two groups are generated and if $\tau_j \in J(R_x, 2) \setminus Aff(R_x, 2)$ and $\lambda_i \in Aff$(R, 2)$\setminus$ $J(R, 2)$ then $\tau_1 \lambda_1 \ldots \tau_n \lambda_n \tau_{n+1}$ does not belongs to Aff(R, 2).

Proof: As noticed above, $T(R_x,2)$ is created by $Aff(R_x, 2)$ and $J(R_x, 2)$.

Supposing

$\tau_1 \lambda_1 \ldots \tau_n \lambda_n \tau_{n+1} = \lambda \in Aff\ (R_x, 2)$ with

$\tau_j \in J(R_x, 2) \setminus Aff(R_x, 2)$ and $\lambda_i \in Aff$(R, 2)$J(R, 2)$for all i.

Then $\tau_1 \lambda_1 \ldots \tau_n \lambda_n \tau_{n+1} \lambda^{-1} = (X, Y)$.

So $bidge$ $(\tau_1 \lambda_1 \ldots \tau_{n+1} \lambda^{-1})$ = (1,1). otherwise it follows from 1.1.2 that $bidge(\tau_1 \lambda_1 \ldots \tau_{n+1} \lambda^{-1})$= $(\prod_{j=1}^{n} \deg \tau_j, \prod_{j=2}^{n} \deg \tau_j)$.

Thus, this bidegree is unequal to (1,1).

since $deg\ \tau_j \geq 2$, a contradiction. (1.1.4)

Note: Let J_0 Become a subgroup of J $(R_x$, 2) the elements are composed of $(X + g(Y), Y)$ Along with $g(Y) \in R_x[Y]$. One easily see that $T(R_x, 2)$is generated by Aff $(R_x, 2)$ & J_0. Though $T(R_x,\ 2)$ It is not the free amalgamated result of $Aff\ (R_x, 2)$ and J_0 over their intersection: For instance, $\tau_1 = (X - Y^2, Y), \lambda_1 = (X, Y + 1)$ and $\tau_2 = (X + (Y + 1)^2, Y)$. Then $\tau_1 \lambda_1 \tau_2 = \lambda_1 \in Aff\ (R, 2)$ but none of $\tau_1, \lambda_1, \tau_2$ fit in to $Aff\ (R_x, 2)$ and J_0 intersection.

***Remark* 1. 1. 5**:

Consider R isn't a domain$T\ (R_x,2)$ It is not the free amalgamated result of $Aff\ (R_x, 2)$ and J $(R_x$, 2) over their intersection.

***Corollary* 4.1.6:** Let
$F1 = (F_1, F_2) \in T(R_x, 2)$ with $bideg\ F = (d_1, d_2)$. Let h_1 Refer to the homogeneous components of $F_1\ f$ degd_1.
Then $d_1|d_2\ or\ d_2|d_1$.
if deg F >1, then we have
If$d_1 < d_2, then\ h_2 = ch_1{}^{d_2/d_1}$, for some c $\in R_x$.
If$d_2 < d_1, then\ h_1 = ch_2{}^{d_1/d_2}$, for some c$\in R_x$.
$If\ d_1 = d_2$, then there exists $\lambda \in Aff(R, 2)$ such that
$\deg F_1' > \deg F_2', where\ (F_1', F_2') := \lambda \circ F$.
Proof: By 1.1.2 we have $F = \lambda_0 \tau_1 \lambda_1 \dots \tau_\ell \lambda_\ell$ with
$bidge\ (\tau_1 \lambda_1 \dots \tau_\ell \lambda_\ell) =$ $(\prod_{j=1}^{\ell} \deg \tau_j, \prod_{j=2}^{\ell} \deg \tau_j)$. Which provides a) Furthermore b) go alone with considering 1), 2) or 3) according to
$a_0 = 0, d_0 = 0\ or\ a_0 d_0 \neq 0$ where $\lambda_0 = (a_0 X + Y b_0 + c_{0,} d_0 X + e_0 Y + f_0)$.

***Remark* 1.1.7:** Indication as in 1.1.6. If $d_1 = d_2$ Next overall, there is no c $\in R$ so that $h_1 = ch_2$ or $h_2 = ch_1$.
***Remark* 1.1.8**.: Consider R the argument is not a domain, a) of 4.1.6 is untrue. Monitor that consider $F1$ is a 1) (resp. 2)) of 1.1.6, then
$tdeg\tau^{-1}\ o\ F1 < tdegF1$, Where $\tau = (X, Y + cX^{d_1/d_2})$ (resp. $\tau =$ (X+$Y^{d_1/d_2}, Y$))).
So, we are having: Process to Choose if $F = (F_1, F_2) \in R_x[X, Y]^2$ belongs to $T\ (R_x, 2)$.
Key: $F = (F_1, F_2) \in R_x[X, Y]^2$.
1. Let (d_1, d_2) = bideg (F_1, F_2).
2. If $d_1 = d_2 = 1$, go to 7.
3. If $d_1 \neq d_2$, go to 5.
4. If there exists $\lambda \in Aff(R_x, 2)$with
tdeg $\lambda F < tdeg\ F$, replace by λF and go to 1, else stop:
$F \neq T(R_x, 2)$.
5. If $d_1 < d_2$, replace F by (F_1, F_2).
6. If $d_1|d_2$ and there exists c $\in R_x$ with $ch_1{}^{d_1/d_2}$, replace $F1$ by (X,Y- cX^{d_1/d_2}) $o\ F$ and to 1, else stop : $F \notin T(R_x, 2)$.
If det $JF1 \in R_x{}^*, stop : F$ $\in$ T $(R_x$,2), else stop : $F \notin T(R_x, 2)$.
7. If $det\ JF\ \in R^*$, stop $F \in T(R, 2)$, else stop:
$F \notin (T, 2)$.
Now we will use this process, or 1.1.6 to tackle the following questions.
Question: According to which condition on R_x is every $R_x -$ automorphism of
$R_x\ [X, Y]$ tame, i.e. According to which condition on R_x do we have the equal to
T $(R_x$,2) = $Aut_{R_x}\ R_x[X, Y]$?

***Proposition* 1.1.9:** $If\ R_x$ is non field, then $T\ (R_x$,2) $\neq Aut_{R_x}\ R_x[X, Y]$.
More precisely let $0 \neq z \in R_x$. Consider being a non-unit $F = (X - 2Y(zX + Y^2) - z$ (zX $+ Y^2)^2, Y + z($zX $+ Y^2))$.
Then $F \in Aut_R\ R_x\ [X, Y] \backslash T(R_x, 2)$.
Proof: Let us consider $F = expD$, here D is the Locally nilpotent Derivation.

$$D = (\text{zX} + Y^2)(-2Y\partial_X + z\partial_Y) on\ R_x[X, Y].$$

$$So\ F \in Aut_{R_x}\ R_x\ [X, Y].$$

$If\ F \in$ T$(R_x$,2) then by 1.1.6 1) we get that - zY^4 =c(z$Y^2)^2$ for some $c \in R_x$.
Therefore $-z =$ cz^2, Or to divided by $-z$ we are having z is a unit in R_x, here we get a contradiction.
***Remarks* 1.1.10**: Automorphism of the $F1$ demonstrated in 1.1.9 is the known as "Nagata automorphism", and introduction by "Nagata" in [78]. It is fascinating to remember that "Nagata" defied his compositional automorphism
$F = \sigma_1{}^{-1}\ \sigma_2 \sigma_1$, where $\sigma_1, \sigma_2 \in Aut_K\ K\ [X, Y]$
(K := the quotient field of R_x) are defined by
$\sigma_1(X, Y) := (X + z^{-1}Y^2, Y),\ \sigma_2(X, Y) := (X, Y + z^2 X)$.
Now the noteworthy point is that the requisite condition is "R_x is a field" It is also appropriate for freedom to have $Aut_{R_x}\ R_x\ [X, Y] = T(R_x, 2)$. That is to quote,

Theorem 1.1.11 ("Jung, van der Kulk"): If k1 is a field, then
$Aut_{k1}\ k1\ [X, Y]$ = T(k1,2). More particularly, $Aut_{k1}\ k1\ [X, Y]$ is the amalgamated free multiplication of $Aff\ (k1$,2) and $J\ (k1$,2) Above their junction.
Considering the strong version of the "Rentschler's theorem" is the basis for the proof we can provide below. The considered Proof of 1.1.11 thus just fits for the case char $k1 = 0$.
We apply to the articles for the general situation, [10] of "Van der Kulk",[11]. Now onwards, $k1$ denotes a field with zero characteristics.

***Theorem* 1.1.12("Rentschler")**:
Considering $0 \neq D1$ as a Locally nilpotent Derivation on k1 $[X_1, X_2]$.
Afterward there exists
$h \in T(k,2) and\ f(X_2) \in k1[X_2]\ s.t$
$h^{-1} D1 h = f(X_2)\partial_{X_1}$.
Let us explain how the "Jung-van der Kulk" theorem is inferred before we prove this outcome.
***Proof Of* 1.1.11**: write A := k1 $[X_1, X_2]$, 1.1.3 we only need to show that $T(k1,2) = Aut_{k1}\ A$. Therefore let $F= (F_1, F_2) \in Aut_{k1} A$. Then $\frac{\partial}{\partial F_1}$ is Locally nilpotent, so by 1.1.12 $\exists h \in T(k1,2)$ & $f(X_2) \in k1[X_2]\ s.t\ h^{-1}\frac{\partial}{\partial F1_1} = \mathrm{f}(X_2)\partial_{X_1}$.
Considering $g \in A$,
Then $h^{-1}\frac{\partial}{\partial F1_1} h(g) = 0$ if $h(g) = 0$
If $h(g) \in ker\frac{\partial}{\partial F_1} = k[X_2],\ i.e. ker h^{-1}\frac{\partial}{\partial F_1} h = k[h^{-1}(F_2)]$ & $\ker f(X_2)\partial_{X_1} = \mathrm{k1}[X_2]$,
So $k1[X_2] = k1[h^{-1}(F_2)] \Rightarrow h^{-1}(F_2) = cX_2 + d,$ for some $c \in k1^*$ and $\mathrm{d} \in k$.
So $F_2 = ch(X_2) + d$.
Moreover $(h^{-1}\frac{\partial}{\partial F1_1} h)(\ h^{-1}(F1_1) = 1$,
$f(X_2)\partial_{X_1}(h^{-1}(F_2)) = 1$ that implies
$\partial_{X_1}(h^{-1}(F_2)) \in k^*, so\ h^{-1}(F_2)c'X_2 + d'(X_2)$
for some $c' \in k1^*\ and\ d'X_2 \in k1[X_2]$.
Consequently $F_2 = c'h(X_2) + d'(h(X_2))$.
Summarizing
$(F_1, F_2) = (c'h(X_2) + d'(h(X_2)), ch(X_2) + d)$.
Since $\mathrm{h} \in T(R_x,2)$ we can prove that $F1 \in T(R_x, 2)$ which finishes the Proof.
Before we prove "Rentschler's theorem", giving certain simplifications regarding "Derand gradings". The following consequence is the beginning.

***Proposition* 1.1.13:**
Considering $R_x = \bigoplus_{m\in Z} R_{x_m}$ as a ring with graded and $D1$ a non-zero Derivation on R_x.
Assuming that $D1$ the following form can be written As a finite amount of derivation of
$D1 = D1_p + D_{p+1} + \ldots + D1_d$
so that $D1_n R_{x_m} \subset R_{x_{n+m}}$, for every $n, m \in \mathbb{Z}$. Consider $D1$ is Locally nilpotent so is $D1_d$.
Proof: It's enough to prove that for each m $\in \mathbb{Z}$ every component of R_{x_m} is described by some power of $D_d. So$ let g $\in R_{x_m}$. Hence $D1$ is Locally nilpotent, There is N in such a way that there is $D1^N g = 0$. The constituent of $D1^N g$ which $\in R_{x_{m+Nd}} = D1_d^N g$. Hence $D1^N g = 0$ it implies that $D1_d^N g = 0$, as required.
From now onwards we limit to the case
$R_x := k1[X] = k1[X_1, \ldots, X_2]$.
For Any single non-zero vector w in $\mathbb{Z}^n$ By defining for each, we combine the w-grading on R d $\in \mathbb{Z}\ R_{x_d}(w)$ similarly k1-vector R area created by all monomials X^a along with $\langle a, w\rangle$ = d .Here $\langle , \rangle$ indicated the standard inner multiplication on ${R_x}^n$. To avoid the confusion, we can use R_{x_d} for the $R_{x_d}(w)$.
For applying above 1.1.13 proposition we show

***Proposition* 1.1.14:**
Considering $0 \neq w \in \mathbb{Z}^n$ and assume on R the w-grading. Considering $D1 \neq 0$ Der on R_x. Afterward we can write $D1$ as a finite addition of derivation
$\sum D1_p$ such that $D1_d R_{x_d} \subset R_{x_{p+d}}$ for all $p, d \in \mathbb{Z}$.
Proof:
The exclusivity follows from the definition of Derivation, therefore we only must show the presence.
So, let $T := \mathrm{c}X^a\partial_i$ be a term appearing in $D1$. substitute $s := a - e_i,$ where e_i denotes the $\mathrm{i-th}$ standard basis vector of $\mathbb{R}^n$.
Then $T(X^m) \in k1\ X^{m+s}\ \forall\ m$. Calling s the strength of T and substitute ea finite decomposition $\mathrm{supp}\, \mathrm{D}1 = \{s \in \mathbb{Z}^n\ |D1 \text{ contain a term of strength s}\}$.
Take $D1(s)$ the sum of all the above set.
Therefore, we get
$\mathrm{D1_p} := \sum_{\langle \mathrm{s,w}\rangle = \mathrm{p}} \mathrm{D1(s)}$.
Obviously $D1 = \sum \mathrm{D1_p}$, since every $\mathrm{non-zero}$ term seeming in $D1$ had certain asset, known as $\mathrm{s_0}$ and therefore $\in \mathrm{D1_{p_o}}$ here $\mathrm{p_0} = \langle \mathrm{s_0, w}\rangle$. So it remains to show that $\mathrm{D1_p} R_{x_\mathrm{d}} \subset R_{x_{\mathrm{p+d}}}$ for every p , d or equal to D1(s) $X^m \in R_{x_{\mathrm{p+d}}}$ for all $X^m \in$

R_{x_d} and all s ∈ $\mathbb{Z}^n$ with ⟨s, w⟩ = p. To see this first, observe that $D1$(s) X^m ∈ $k1$ X^{m+s}, we demonstrate the $\langle m+s,w\rangle$= $p + d$.
But since then, this conveniently follows $\langle m+s,w\rangle = \langle m,w\rangle + \langle s,w\rangle = d+p$ using that $X^m \in R_{x_p}$.
The decomposition of D1 given in 1.1.13 is called w-homogeneous decomposition of $D1$. Consider p is max with $D1_p \neq 0$, p is known as w –deg of $D1$, denoted $wdegD1$. In case $w = (1,1,\ldots,1)p$ is known as deg of $D1$ and indicated by $degD1$. The following proposition plays an important part in getting proof of "Theorem of Rentschler's".

***Proposition* 1.1.15**: Considering $R_x = \bigoplus_{d\in Z} R_x$ any grading on it will be R_x = k1[X], therefore It doesn't even need to be a $w-$ grade, and Considering $D1 = \sum D1_p$ not an infinite decomposition sustaining $D1_p R_{x_d} \subset R_{x_{p+d}}$ $\forall p, d \in \mathbb{Z}$.
Assuming p be greatest with $D1_p \neq 0$.
If $D1_p = D1(s)$ for certain s = $(s_1,\ldots,s_2) \in \mathbb{Z}^n$, where all $s_1 \geq 0$, then $D1$ is not LN.
Proof : Considering ∈ $\mathbb{Z}^n$. We from following remark that D1(s) = $X^s \sum c_j X_j \partial_j$ for some $c_j \in k$. Since $D1(s)$ = $D1_p \neq 0$ J exists with the $c_j \neq 0$. Observing that if all $s_i \geq 0$, then $D(s)(X_j) = c_j X^s X_j$ implies that $c_j X^s$ is a positive eigenvalue of
D1 (s)in k1[X]. Thus, from 1.8.32 $D1(s)$ is not Locally nilpotent, i.e. $D1_p$ is not Locally nilpotent Before applying 1.1.13 Consider the condition $n = 2$ and write R_x = k1[X,Y] in its place of k1[X_1, X_2]. Considering $D1 \neq 0$ be a Der on R_x. Then denoting an element of supp $D1$ as (s, t). Thus s, t ≥ −1 and e.g. $(s,-1) \in supp$ $D1$ That implies that $D1$ includes a term of the form c$X^s\partial_j$ with c ∈ k^*.

***Corollary* 1.1.16**:
Consider $D1$ is Locally nilpotent, then either
$D1$ = f′(Y) ∂_X, for some f′(Y) ∈ $k1[Y]$ or
$D1$ = f′(X) ∂_Y, for some f′(X) ∈ $k1[X]$ or
there exists $s_0, t_0 \geq 0$ such that $(s_0,-1)$ and (-1 , t_0) belong to supp D1 and furthermore supp D1 is connected in the triangle with vertices $(s_0,-1)$, $(-1,-1)$, $(-1, t_0)$.
Proof : Let $p_0 := \deg D1$. So $p_0 \geq -1$. If $p_0 = -1$ then $D1$ = a∂_X + b∂_Y for few
$a, b \in k1$, both not zero and we're completed with the proof. Therefore, now considering $p_0 \geq 0$ and Considering ℓ indicated the line $x + y = p_0$
Case I: ℓ includes a point of the method $(-1, t_0)$. By description of ℓ all the points of a point supp D1 is on or under this axis. Turn l around the dot now $(-1, t_0)$ (clockwise) until one hit another point of supp D1. Consider ℓ about more than one supply point $D1$ We don't have to turn $\ell, i.e.$ we change it over an angle of degree zero. In case one does not meet any other point of supp D1, afterward clearly
supp D1 = {(-1 , t_0)} and consequently $D1$ = c$Y^{t_0}\partial_X$ for few c ∈ $k1^*$and we are in case 1). If one only meets points of the form (-1, t) then all these points satisfy t ≤ t_0.Hence Even, we are in the condition 1). We should then conclude that we have at least one segment $(s,t) \in$ supp $D1$ with $s \geq 0$. Given We have these points to select second coordinate minimal. Label this point (s',t'). We demonstrated that $t' = -1$, what substitute in case 3). Hence assume that $t' \geq 0$. we will come across a contradiction. We know that $s' \geq 0, so$ together $s', t' \geq 0$. Now move ℓ over a tiny angle around (s',t') (clockwise) we notice that a line appears,ℓ' : $\omega_1 x + \omega_2 y = d$ with $\omega_1, \omega_2, d \in \mathbb{Z}$ such that (s',t') is the solitary point of supp $D1$ On this line, and in addition, all other points of supp $D1$ are under ℓ'. substituting $\omega := (\omega_1, \omega_2)$ and $p := \omega \deg D1$, it follow that $D1_p = D1((s',t'))$. Applying 1.1.15 demonstrate that $D1$ is not Locally nilpotent, a contradiction. Hence $t' = -1$.
CASE II: ℓ Include a point in the form of $(s',-1)$. This condition is same as case I.
CASE III: ℓ Include either a point in the form of $(s,-1)$ or a point in the form of
$(-1,t)$.Considering $(s, t) \in \ell$ with t min. Hence $s, t \geq 0$ We will see a line once again,ℓ' so that (s,t) the single point of the supp D1 on ℓ' and entirely other points of supp D1 are under ℓ'. Next, as over, we are having a negation, that summarises the proof.

Proof Of "Rentscher's Theorem"
1) Let's use 1.1.16 & consider $D1 = f(X)\,\partial_Y$ take $h = (Y,X)$.
Afterward h^{-1} $D1h = f(Y)\,\partial_X$. Therefore we consider that we are having s_0, t_0 in 4.1.16. We can write $s_0(D1), t_0(D1)$ in its place of s_0, correspondingly t_0 if essential.
Considering ℓ'be the line crossing through both $(s_0,-1)$ and (-1, t_0). Hence ℓ' is provided by the eq $(t_0+1)x$ + $(s_0 + 1)y = p$, where $p = s_0 t_0 - 1$.
Considering $\omega := (t_0+1, s_0+1)$.
Therefore by ω $degD = p$ and by 1.1.13
$D1_p$ is Locally nilpotent. Inscribe
$D1_p = gD1_1$ $where$ $D1_1 = a\partial_X + b\partial_Y$
fulfils $gcd(a,b)$ = 1. By $D1_1$ is Locally nilpotent and $D1_1(g) = 0$. since $D1_p$ is w-homogenous .
2) Since $D1_1$ is Locally Nilpotent with
$gcd\ (a,b) = 1$ it tracks from that $D1_1$ has a piece of it in the $k1[X,Y]$,

Thus in specific $a(0) \neq 0$ or $b(0) \neq 0$.
We may assume that
$a(0) \neq 0$. So $D1_1$ contain a term of the form $c\partial_X$ with $c \in k^*$.
Since $(s_0, -1) \in$ supp $D1_p$, i.e. $D1_p$ contain a. The concept of the form $c' X^{s_o} \partial_X$ and since
$D1_p$ = g$D1_1$ it follows that $D1_1$ *also* contains a concept in the form $dX^r\partial_Y$ with $r \geq 0$ and
d ∈ $k1^*$. But then follows that $D1_1 = c\partial_X + dX^r\partial_Y$, namely $D1_1$
$w-$ homogenous whence supp $D1_1$ It's on a line that is, Obviously, the line that goes through $(-1,0)$ and $(r,-1)$. This line, however, does not contain any other integer points.
co-ordinates ≥ -1.
3)since $D1_1(g) = 0$ we get that g ∈ ker
$D1_1 = k[Y - \frac{d}{(r+1)c} X^{r+1}]$.
The homogeneity of g implies that
g = a$(Y - \frac{d}{(r+1)c} X^{r+1})^N$ for some
a ∈ $k^*$and N ∈ ℕ.
So $D1_p$ =a$\left(Y - \frac{d}{(r+1)c} X^{r+1}\right)^N$ $(c\partial_X + dX^r\partial_Y)$.
4) Lastly Considering h be the automorphism provided by $h(X) = X$
and $h(Y) = Y - \frac{d}{(r+1)c} X^{r+1}$.
Then one easily verifies that $h^{-1}D1_p h = aY^N c\partial_X$. so single effortlessly demonstrated that
$h(R_{x_d}) \subset R_{x_d}$ and therefore $h(R_{x_d}) = R_{x_d}$ for each d ∈ ℤ. Accordingly
$(h^{-1}D1h)$m = $h^{-1}D1_m h$ ∀m ∈ ℤ. Particularly the $w-$ degree of $h^{-1}D1h$ is equality to
$\omega-$ degree of D1 which equal to p. Therefore $h^{-1}D1_p h$ = aY^N $c\partial_X$ For a point in the form, there is no contribution $(s,-1)$in supp $h^{-1}D1h$ impending from $(h^{-1}D1h)_p$. Consequently $s_0(h^{-1}D1h) < s_0(D1)$. The theorem then proceeds with induction on $s_0(D1) + t_0(D1)$.

***Remark* 1.1.17**: Further proof of equality $Aut_{k1}k1[X,Y] = T(k1,2)$ is given in the next section of this section, see 1.3.6. This proof is a result of the "Abhyankar-Mohs" theorem.
As result of we get some information about a co-ordinate in the above proof k1[X, Y].
First if $0 \neq f = \sum f_{ij} X^i Y^j \in k1[X,Y]$ we put $supp\ f := \{(i,j) \in \mathbb{Z}^2 | f_{ij} \neq 0\}$.
The "Newton polygon" of , represented as $N(f)$, is the convex hull of $supp\ f$ ∪ {0,0}. Lastly saying that f *has* $r \geq 1$ points on infinity if f_+, the highest deg homogeneous part of f giving to the
$(1,1)-$ grading has r distinct (linear) prime factors in $k1[X,Y]$.

***Proposition* 1.1.18**: *If* f is a co-ordinate in $k1[X,Y]$ then
1) $N(f)$ is a triangle with vertices (0,0) , $(n, 0)$, $(0, m)$, where n, m are integers ≥ 0.
2) f has one point at infinity.
Proof: 1) Since f is a coordinate there exists g ∈ $k1[X,Y]$ s.t $k1[X,Y] = k1[f,g]$ and
det $(f,g) = 1$.
Hence $D1 := \frac{\partial}{\partial_g}$ is Locally nilpotent on $k1[X,Y]$ and
$D1 = f_Y\partial_X - f_X\partial_Y$. Using this description of $D1$ we get that
$(i,j) \in suppf$ if $(i-1, j-1) \in supp\ D1$.
Using 1.1.16 We discover that a triangle is $N(f)$. 2) By 1) We realise that a triangle is $N(f)$.
consider $n > m$ then $f_+ = c_1 X^n$ for some $c_1 \in k1^*$ and if $n < m$ clearly
$f_+ = c_2 Y^m$ with $c_2 \in k^*$. Lastly, if $n = m$, afterward giving to the (1,1)-grading we get $D1_p = f_{+Y}\partial_X - f_{+X}\partial_Y$, here p is the (1,1)-deg of $D1$.
Since from the proof of above Theorem, we see that
$D1_p$ =a$\left(Y - \frac{d}{(r+1)c} X^{r+1}\right)^N$ $(c\partial_X + dX^r\partial_Y)$.
for some $a, c, d \in k^*, N \geq 0$ *and* $r \geq 0$.
Also Y$-\frac{d}{(r+1)c} X^{r+1}$ is (1,1)-homogeneous.
So, r = 0 and consequently and
$D1_p$ =a$\left(Y - \frac{d}{c} X\right)^N$ $(c\partial_X + d\partial_Y)$.
Since , as observed above, $D1_p = f_{+Y}\partial_X - f_{+X}\partial_Y$ we obtain
$f_+ = ac\frac{1}{N+1}\left(Y - \frac{d}{c} X\right)^{N+1}$. Hence the Proof.
Remarks: 1) Le R'_x be a CR that is non integral domain is, considering $ab = 0$ here a and b these are positive components in R'_x.

a) Letting $\tau := (X + aY^2, Y)$ and
$\lambda := (X, Y + bX)$. Then both τ and λ do not belong to $Aff\ (R'_x,2) \cap J(R_x, 2)$ and that $\tau\ \lambda\tau^{-1} = \lambda$.
So T(R'_x,2) is not the free outcome of $Aff(R'_x,2)$ and J(R'_x,2) on their intersection.
b) Let $F = (X, Y + aX^2)\ o(X + aY^2, Y)$.
Then
$F = (X + aY^2, Y + aX^2)$, So $(d_1, d_2) := bideg\ F = (2,3)\ and\ d_1$ does not divide d_2.
2)suppose that $R1_x$ is a domain which is not a field and $z \in R1_x$ is such that both z and $1 - z$ are nom-units is $R1_x$.
Put F = $\tau\ \lambda, where\ \lambda = ((1 - z)X + zY, -zX + (1 - z)Y)$ and = (X+Y^2, Y).
a)Then $\lambda \in Aff(R1_x,2)$,
F $\in T(R1_x, 2) and\ bideg\ F = (2,2)$.
b) Let $h_1 h_2$ be as in 1.1.6. Then there does not exist c $\in R1_x$ such that $h_1 = ch_2$ or $h_2 = ch_1$.
3) Letting z a component in R_x and F1 the "Nagata automorphism" defined in 1.1.9. Then that $F \in T(R_x, 2)$.
4)Let R_x be a ring encompassing the element a, b, Assuming $f(T) \in R_x$ [T] and define
F := (X+ bf(aX+bY), Y-af(aX+bY)). a) Then F1 $\in Aut_{R_x} R_x[X, Y]$. [we get this from D1 $:= f(aX + bY)(b\partial_X - a\partial_Y)$ is a Locally nilpotent Derivation on R_x [X,Y] and that F = expD1.] b) Now assume that R_x is a domain such that R_xa +R_xb is not a principal ideal. Consider $degf(T) \geq 2$, then $F1 \notin T(R_x, 2)$.

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