An Edge Z-Algebra

Nesrin. Alhmesh[1*], Haytham. Arabi[2,] and Safwan. Aouira[3]

Abstract
We introduce- in this paper- the notion of edge Z-algebra and investigate its properties. Also we introduce the notion of edge Z-subalgebra, and show the relation between these two notions. And we introduce the notion of Z-closed set, and re-introduce the notion of Z-subalgebra, and show the relation between these two notions. Then we prove that the union of Z-closed set with Z-subalgebra is Z-subalgebra in edge Z-algebra.

Keywords: Z-algebra, edge Z-algebra, Z-Subalgebra, edge Z-subalgebra, Z-closed set.

[1*]Aleppo University, Faculty of Science, Department of Mathematics, Aleppo, Syria
[2]Aleppo University, Faculty of Science, Department of Mathematics, Aleppo, Syria
[3]Aleppo University, Faculty of Science, Department of Mathematics, Aleppo, Syria

***Corresponding Author:** Nesrin Alhmesh;
[*]Aleppo University, Faculty of Science, Department of Mathematics, Aleppo, Syria Email address: nesrin_alhmesh@alepuniv.edu.sy Orcid no: 0009-0002-1519-6950

1. Introduction

Algebraic structure is an important branch of mathematics, it has a lot of applications in many fields as computer science, information science, and coding theory, among others. The concept of Bck-algebra had been introduced by Y. Imai and K. Iseki in 1966, after that both of them introduced the class (BCI-algebra) as a generalization of Bck-algebra, in 1980 [1]. In 1999, the concept of d-algebras had been introduced by J. Neggers and H.S. Kim [2]. Since then, several interesting generalizations of BCK\BCI-algebras have been presented and researched ([5], [8]).

M. Chandramouleeswaran, P. Muralikrishna, K. Sujatha, and S. Sabarinathan construct the concept of Z-algebra. They gave us the proof: that Z-algebra is not a generalization of BCK/BCI-algebras, that Z-algebra is a different concept from other abstract algebras such BE-algebras, BF-algebras, d-algebras and so many else- 2017 [4]. Since that time, several papers have been published examining the Z-algebra ([9], [10], [11]). Our paper is a follow-up to these works.

2. Preliminares:

we review the concepts of BCK-algebra and d-algebra. we mention the definition of Z-algebra and some properties that are needed for our work in this part.

Definition 1 (see [2]). let X be a nonempty set with constant 0 and let * be a binary operation since the following axioms hold:

$$(I)\quad x * x = 0$$
$$(II)\quad 0 * x = 0$$
$$(III)\quad x * y = 0\ \&\ y * x = 0\ \Rightarrow x = y\quad \forall\, x, y \in X$$

Then $(X, * ,0)$ is called a d-algebra.

- If $(X, * ,0)$ is a d-algebra with additional axioms hold:

$$(IV)\quad ((x * y) * (x * z)) * (z * y) = 0$$
$$(V)\quad (x * (x * y)) * y = 0\ \ \forall\, x, y, z \in X$$

Then $(X, * ,0)$ is called a Bck-algebra.

Definition 2. (see [4]). let X be a nonempty set with constant 0 and let * be a binary operation since the following axioms hold:

$$(Z_1)\quad x * 0 = 0$$
$$(Z_2)\quad 0 * x = x$$
$$(Z_3)\quad x * x = x$$
$$(Z_4)\quad x * y = y * x \text{ when } x \neq 0\ \&\ y \neq 0\ \ \forall\, x, y \in X$$

Then $(X, * ,0)$ is called a Z-algebra.

Example 1. (see [4]). Let we have a set $X = \{0,1,2,3\}$, a constant 0 and a binary operation * with the Cayley's table:

*	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

(1)

Then, $(X, * ,0)$ is a Z-algebra.

Definition 3. (see [2]). If $(X, * ,0)$ is a d- algebra and $x \in X$. Define the set $x * X = \{x * a;\ a \in X\}$. If $* X = \{0, x\}\ \forall x \in X$, then X is said to be edge.

Definition 4. (see [3]) Let X be a d-algebra and I a nonempty subset of X. if $x * y \in I, \forall x, y \in I$, then I is called a d-subalgebra of X.
It's clear that the constant "0" belongs to every d-subalgebra "I". Because: for any x in I, we have $0 = x * x \in I$.

Definition 5. (see [4]) Let X be a Z-algebra and I a nonempty subset of X. if $x * y \in I, \forall x, y \in I$., then I is called a Z-subalgebra of X.

Example 2. (see [4]) In the example 1. Two Z-Subalgebras of X were given: $A = \{1,3\} \subset X$ and $B = \{2,3\} \subset X$. and another subset $C = \{1,2,3\} \subset X$ which is not a Z-Subalgebra of X was given.

3. Main Results:
Definition 6. let $(X,* ,0)$ be a Z- algebra and $x \in X$. Define:

$$X * x = \{a * x \,; a \in X\} \qquad (2)$$

X is said to be edge if for any x in X,

$$X * x = \{0, x\} \qquad 3)$$

Example 3. Let we have a set $X = \{0,1,2,3\}$, a constant 0 and a binary operation * with the Cayley's table:

*	0	1	2	3
0	0	1	2	3
1	0	1	0	0
2	0	0	2	0
3	0	0	0	3

(4)

Then, $(X,* ,0)$ is a Z-algebra. We can easily see that $(X,* ,0)$ is an edge Z-algebra, since (3) holds for all x in X.

Example 4. Let we have a set $X = \{0,1,2,3,4\}$, a constant 0 and a binary operation * with the Cayley's table:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	0	4
2	0	2	2	1	0
3	0	0	1	3	1
4	0	4	0	1	4

(5)

Then, $(X,* ,0)$ is a Z-algebra. We can easily see that

$$X * 1 \neq \{0,1\} \qquad (6)$$

Because

$$2 * 1 = 2 \notin \{0,1\} \qquad (7)$$

then $(X,* ,0)$ is non-edge Z-algebra.

Theorem 1. let $(X,*,0)$ be an edge Z- algebra, then for all x and y in X, we have:

$$y * x = \{x \;\; if\; y = x \;\; or \;\; y = 0 \; 0 \qquad otherwise \qquad (8)$$

Proof: let $x, y \in X$. Then, then we notice definition 2 and find:
1. if $y = x$, we have

$$y * x = x * x = x \qquad (9)$$

2. If $y = 0$, we have

$$y * x = 0 * x = x \qquad (10)$$

3. Otherwise, if $x \neq y$ and $y \neq 0$.

If $x = 0$, we have

$$y * x = y * 0 = 0 \qquad (11)$$

If $x \neq 0$ and we have $y \neq 0$, X is an edge Z- algebra, we have

$$x * y = \{0, y\} \ \& \ y * x = \{0, x\} \qquad (12)$$

If $y * x = 0$, complete the proof. If $y * x = x$, then

- If $x * y = 0$, then $0 = x$, a contradiction.
- If $x * y = y$, then $y = x$, a contradiction too.

❖ We will call that

$$X^* = X - \{0\}. \qquad (13)$$

Proposition 1. let (X, $*$,0) is an edge Z- algebra, then:

$$1.\ x = y \Longrightarrow x * y = y * x = x \ \forall x, y \in X$$
$$2.\ x \neq y \Longrightarrow x * y = y * x = 0 \ \forall x, y \in X^* \qquad (14)$$

Proof: we notice definition 2 and see,

$$1.\ x = y \Longrightarrow x * y = y * x = x * x = x$$

2. we notice (12),

- If $x * y = 0$ and $y * x = 0$, then $x * y = y * x = 0$, complete the proof.
- If $x * y = y$ and $y * x = 0$, then $y = 0$, a contradiction.
- If $x * y = 0$ and $y * x = x$, then $0 = x$, a contradiction.
- If $x * y = y$ and $y * x = x$, then $y = x$, a contradiction too.

Proposition 2. let $(X, *, 0)$ be a Z- algebra, then X is an edge Z- algebra if, and only if,

$$x * X^* = \{0, x\} \ \forall x \in X^* \qquad (15)$$

Proof:

First: Assume that X is an edge.

$$\text{let } x * y \in x * X^*;\ x, y \in X^*$$

1. Since $x \neq 0$ and $y \neq 0$, then $x * y = y * x$ (by (Z_4)).

Now we have:

$$x * y = y * x \in X^* * x \subseteq X * x = \{0, x\}$$
$$\Rightarrow x * y \in \{0, x\} \ \forall x * y \in x * X^*$$
$$\Longrightarrow x * X^* \subseteq \{0, x\} \qquad (16)$$

2. if $x \neq y$, we notice (14), then we have

$$x * y \in x * X^*$$
$$\Longrightarrow 0 \in x * X^* \qquad (17)$$

if $x = y$, we notice (Z_3), then

$$x * y \in x * X^*$$
$$\Longrightarrow x \in x * X^* \qquad (18)$$

By (17) and (18) we see $\{0, x\} \subseteq x * X^*$ (19)

By (16) and (19) we find that $x * X^* = \{0, x\} \forall x \in X^*$.

Second: Assume (15) holds, and we notice definition 2.

Let $x, y \in x$

1. Let $y * x \in X * x$;
- If $x \neq 0$ and $y \neq 0$, then $y * x = x * y \in x * X^* = \{0, x\}$ by (15).
- If $x = 0$, then $y * x = y * 0 = 0 \in \{0, x\}$.
- If $y = 0$, then $y * x = 0 * x = x \in \{0, x\}$.

From these three cases we see that

$$X * x \subseteq \{0, x\} \qquad (20).$$

2. If $x = 0$ then $0 = y * 0 = y * x \in X * x$ (by (Z_1)). And we have $x = x * x \in X * x$ (by (Z_3)). Then

$$\{0, x\} \subseteq X * x \qquad (21).$$

By (20) and (21) we find (3), then X is an edge.

Proposition 3. let $(X, * ,0)$ be a Z- algebra. X is an edge Z- algebra If, and only if,

$$x * y = 0 \ \forall x, y \ (x \neq y) \in X^* \qquad (22).$$

holds.
Proof: if X is an edge, and we notice (14), then (22) holds.
If (22) holds, then:

$$\text{Let } y * x \in X * x; x, y \in X$$

- $x \neq 0$ and $y \neq 0$, then $y * x = x * y = 0 \in \{0, x\}$ by (22).
- $x = 0$, then $y * x = y * 0 = 0 \in \{0, x\}$ (by (Z_1)).
- $y = 0$, then $y * x = 0 * x = x \in \{0, x\}$ (by (Z_2)).
- $x = y$ then $y * x = x * x = x \in \{0, x\}$ (by (Z_3)).

Now we find (20).
And if $x = 0$, then $0 = y * 0 = y * x \in X * x$ (by (Z_1)). And we have $x = x * x \in X * x$., then we get (21).
By (20) and (21) we find (3), then X is an edge.

Theorem 2. let $(X, *, 0)$ be an edge Z- algebra, then for all x and y in X, we have

$$(x * y) * x = (y * x) * x = x \qquad (23).$$

Proof: let $x, y \in X$. We notice definition 2, then:

$$x = 0 \ or \ y = 0 \Longrightarrow (x * y) * x = x \ \& \ (y * x) * x = x$$

$$x \neq 0 \ \& \ y \neq 0 \Longrightarrow (x * y) * x = (y * x) * x$$

Since X is an edge Z- algebra, we have $y * x \in X * x = \{0, x\}$.

$$y * x = 0 \ or \ y * x = x \Longrightarrow (x * y) * x = (y * x) * x = x$$

Theorem 3. let $(X, * ,0)$ be a Z- algebra. If

$$(x * y) * x = x \ \forall x, y \in X \Leftrightarrow x * y = 0 \ \forall x, y \ (x \neq y) \in X^* \qquad (24).$$

Proof: let $x, y \ (x \neq y) \in X^*$. Since x≠0 & y≠0, then $x * y = y * x$ (by (Z_4)). Now assume that there are $x, y \ (x \neq y) \in X^*$such that $x * y \neq 0$. Therefore,
$either \ x * y = x, x * y = y \ \ or \ x * y = z; z \in X^*, z \neq x, z \neq y.$
Now

$$x * y = x \Longrightarrow (y * x) * y = (x * y) * y = x * y = x \neq y$$

a contradiction.

$$x * y = y \Longrightarrow (x * y) * x = y * x = x * y = y \neq x$$

a contradiction.

$$x * y = z \Longrightarrow x = (x * y) * x = z * x = x * z$$
$$\Longrightarrow (z * x) * z = x * z = x \neq z$$

a contradiction too.
Then $x * y = 0 \ \forall x, y \ (x \neq y) \in X^*$.

Now, let$x, y \in X$:
- $x = 0$, then $(x * y) * x = (0 * y) * 0 = 0 = x$
- $y = 0$, then $(x * y) * x = (x * 0) * x = x$
- $x = y$, then $(x * y) * x = (x * x) * x = x$
- $x \neq 0, y \neq 0$ and $x \neq y$, then $(x * y) * x = 0 * x = x$, (by $x * y = 0$).

Then $(x * y) * x = x \; \forall x, y \in X$

❖ As a result, we can conclude the following theorem:

Theorem 4. let $(X,* ,0)$ be a Z- algebra, then

X is an edge $\Leftrightarrow x * X^* = \{0, x\} \;\; \forall\, x \in X^*$

$$\Leftrightarrow x \; * \; y = 0 \; \forall x, y \; (x \neq y) \in X^*$$

$$\Leftrightarrow (x \; * y) * x \; = x \; \forall x, y \in X$$

Proof: it is clear from proposition 2, remark 1, theorem 2 and theorem 3.

- In the definition " Z-subalgebra " that is given by definition 5, we notice that the constant "0" – is not necessary- belongs to every Z-subalgebra. Because: for any $x \neq 0$ in X, we have $x * x = x \in I$ (by (Z_3)). Hence $I = \{x\}$ is a Z-subalgebra. We see that (Z_1) and (Z_2) don't hold since$0 \notin I$. that makes a contradiction with concept of substructure, (see [6],[7]).

So we suggest to rename the nonempty subset "I" of a Z-algebra X which is defined in definition 5 as a Z-closed set. And if $0 \in I$, where "I" is Z-closed set then we call "I" a Z-subalgebra of X.
It's clear that every Z-subalgebra of X is a Z-closed set of X, but the converse need not be true in general.

Example 5. Let we have a set $X = \{0,1,2,3\}$, a constant 0 and a binary operation * with the Cayley's table:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	0
2	0	2	2	0
3	0	0	0	3

(25)

Then, $(X,* ,0)$ is a Z-algebra. $I = \{1,2\}$ is a Z-closed set of X, but not a Z-subalgebra of X, since $0 \notin I$.

Theorem 5. The intersection of a family of Z-subalgebra in a Z-algebra X is a Z-subalgebra in X.
Proof: Let $I_k, k \in K$ is a Z-subalgebra of Z-algebra X. If $x, y \in \bigcap_{k \in K} I_k$ then $x, y \in \; I_k$ for all k in K, so $x * \; y \in \; I_k$ (since I_k is a Z-subalgebra for all k in K), so $x * y \in \bigcap_{k \in K} I_k$
In the same way, we proof that the intersection of a family of Z-closed sets in a Z-algebra X is a Z-closed set in X.

Remark 1. let both I, J are Z-subalgebras of X, and let K is Z-closed set of X. then $I \cup J$ &$K \cup J$ are not necessary be Z-subalgebras in X. as the following example.

Example 6. Let we have a set $X = \{0, a, b, c, d\}$, a constant 0 and a binary operation * with the Cayley's table:

*	0	a	b	c	D
0	0	a	b	c	D
a	0	a	a	c	B
b	0	a	b	a	C
c	0	C	a	c	0
d	0	B	c	0	D

(26)

Then, $(X,*,0)$ is a Z-algebra, and it is clear that both $I = \{0,b\}$ and $J = \{0,d\}$ are Z-subalgebras in X. but $I \cup J = \{0,b,d\}$ is not Z-subalgebra in X, since $b * d = c \notin I \cup J$.
And it is clear that $K = \{a,b\}$ is Z-closed set in X, but $K \cup J = \{0,a,b,d\}$is not Z-subalgebra in X, since $b * d = c \notin K \cup J$.
The condition that makes the union of a Z-closed set and a Z- subalgebra be a Z-subalgebra in X, when X is edge Z-algebra, and the following theorem showing that.

Theorem 6. Let I be a Z-closed set and let J be a Z-subalgebra in edge Z-algebra X, then $I \cup J$ is a Z-subalgebra in X.
Proof: $0 \in J$, then $0 \in I \cup J$.
Let $x,y \in I \cup J$.
If

$$x,y \in I \Rightarrow x * y \in I$$
$$\Rightarrow x * y \in I \cup J \qquad (27)$$

If

$$x,y \in J \Rightarrow x * y \in J$$

Then we have (27).
If

$$x \in I, y \in J \Rightarrow x * y \in \{0,y\}$$

Because X is an edge Z-algebra. hence $x * y \in J$, now we get (27).
By the same way we proof that if $x \in J$ & $y \in I$, then we get (27).
We can proof by the same way that the union of two Z-subalgebras in edge Z-algebra X is a Z-subalgebra in X.

Remark 2. Let X be a Z-algebra, then $I = \{0,x\}$ is Z-subalgebra for all x in X.
Proof: $0 \in I, x * 0 = 0 \in I$ (by (Z_1)), and $0 * x = x \in I$ (by (Z_2)).

Definition 7. If $(X,*,0)$ is a Z- algebra and I a Z-subalgebra in X. we call "I" an edge Z-subalgebra in X, if for any x in I, $I * x = \{0,x\}$.

Example 7. Let we have a set $X = \{0,1,2,3\}$, a constant 0 and a binary operation * with the Cayley's table:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	2
2	0	2	2	3
3	0	2	3	3

(28)

Then, $(X,*,0)$ is a Z-algebra. We can easily see that both $I = \{0,1\}$ and $J = \{0,1,2\}$ are Z-subalgebras. We notice that "I" is an edge Z-subalgebra but J is non-edge Z-subalgebra since $2 * 1 = 2 \notin \{0,1\}$.

It is clear that every edge Z-subalgebra in Z-algebra X is Z-subalgebra, but the converse need not be true in general.

4. Conclusions:

To investigate the structure of an algebraic system, it is clear that edge Z-algebras plays an important role, and we developed this concept and studied some of its properties. And we found some equivalent conditions to edge Z-algebra, which are important in studying edge Z-algebra. And we studied the structure of Z-subalgeba which has an importance in studying the properties of Z-algebra.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Reference:

1. K. Iseki, “On BCI-algebras,” Mathematics Seminar Notes, Kobe University,8(1980),125-130.
2. J. NEGGERS and H.S.KIM, “On d-algebras,” Math. Slovaca 49 (1999), 19–26.
3. J. Neggers, Y. B. Jun and H. S. Kim, “On d-ideals in d-algebras,” Math. Slovaca, 49(1999), 243-251.
4. M. Chandramouleeswaran, P. Muralikrishna, K. Sujatha and S. Sabarinathan, “A note on Z-algebra,” Italian Journal of Pure and Applied Mathematics, 2017; 38: 707-714.
5. J. Neggers, S.S. Ahn, H.S. Kim, “On Q-algebras,” Int. J. Math. and Math.Sci., 27 (2001), 749–757.
6. A.Walendziak, “On BF-algebras,” Mathematica Slovaca, vol. 57, no. 2, pp. 119–128, 2007.
7. A.B. Kim, H. S. Kim, “On BG-algebras,” Demonstration Mathematica, 41(2008), 497-505.
8. K. Kalaiarasi and V. Manimozhi, “Fuzzy sets in KM-algebras,” Gedrag and Organisate Review, vol. 33, no. 2, pp. 2034–2046, 2020.
9. S.Sowmiya and P.Jeyalakshmi, “Z-Homomorphism and Cartesian product on Fuzzy α-Translations and Fuzzy β-Multiplications of Z-algebras”, AIP Conference Proceedings 2261 (2020) ,pp.030098-1–030098-5.
10. Nouri, Adel Hashem, and Layth Salah Mahdi. “BS-algebra and Pseudo Z-algebra.” Journal of Physics: Conference Series. Vol. 1963. No. 1. IOP Publishing, 2021.
11. Sowmiya, S., and P. Jeyalakshmi. “On Fuzzy Implicative Ideals in Z-Algebras.” (2022).