Approximate Fixed Point Theorem With Contractive Conditions

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Abstract:
In this article, we have reviewed previous approximative fixed point theorems and developed some new fixed point theorems for entire metric space and for b-metric space for the maps meeting the contractive criteria Banach contraction, U contraction, Reich Contraction, and Kannan Contraction.

Keywords: Fixed point theorem, Approximate fixed point theorem, Contractive conditions, metric space, b metric space, complete b metric space.

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Introduction:

In applied mathematics, fixed point theory provides answers to a variety of issues. Yet, practise has shown that in many real-world circumstances, an approximation will suffice. Hence, the existence of virtually stationary points is required but not strictly required. Another kind of real-world circumstances that result in this approximation is when the requirements that must be met to ensure the existence of fixed points are considerably too stringent in comparison to the actual issue at hand.

Let U be a metric space's self map (A, p). Let's try to get a roughly correct answer to the equation Ua=a. The point q is referred to as an approximation solution to the equation Ua=a, or equivalently, q A is an approximate fixed point of U, if there is a point q A such that p(Uq,q) , where p is a positive number.

In addition to mathematical economics, noncooperative game theory, dynamic programming, nonlinear analysis, variational calculus, the theory of integro differential equations, and several other analytical fields ([2], [4], [5], [9], [10]), the theory of fixed points and, consequently, approximate fixed points have applications in these fields as well.

By applying the Brouwer fixed point theorem to a discontinuous map, Cromme and Diener [3] discovered approximate fixed points. Hou and Chen's [6] results have been expanded to set valued maps. Espinola and Krik [5] have achieved intriguing outcomes in the product space. By reducing the requirements on the spaces, Tijs et al. [10] have presented approximate fixed point theorems for contractive and non-expansive maps. These findings were later expanded by R. Branzei et al. [2] to multifunctions in Banach spaces. For operators satisfying criteria of the type posed by Kannan, Chatterjea, and Zamfirescu on metric spaces, M. Berinde [1] has discovered approximative fixed point theorems.

This article's primary goal is to define certain approximative fixed point results in metric spaces under various contractive conditions.

I.Preliminaries:

Definition 1.1:

If (A, p) is a metric space and U: $A \to A$. Let ε be a positive number. Then a point $q \in A$ is an ε fixed point of U if $p(Uq,q) \leq \varepsilon$. A map U: $A \to A$ is said to have approximate fixed point property if, for each $\varepsilon > 0$, the map A possesses at least one ε fixed point.

Definition 1.2:

The set of all ε fixed points of U for a givenε, is defined as below

$T(U) = \{q \in U: q \text{ is a } \varepsilon \text{ fixed point of } U\}$.

Definition 1.3:

If U: $A \to A$, then U has the approximate fixed point property if $\forall \varepsilon > 0$, $T(U) \neq \phi$.

Definition 1.4:

If (A,p) is a metric space, U,V : $A \to A$ then V is said to be U asymptotic regular if $p(UV^m(q), UV^{m+1}(q)) \to 0$ as $m \to \infty$, $\forall q \in A$.

Definition 1.5:

A self map U: $A \to A$ on metric space (A, p) is said to be subsequentially convergent if we have, for every sequence $\{r_m\} \in A$, if $\{U r_m\}$ is convergent then $\{r_m\}$ has a convergent subsequence.

Lemma 1.6[8]:

If (A,p) is a metric space and U,V : $A \to A$ be two commuting maps. If V is U asymptotically regular, then V has an approximate fixed point property.

II. Main Results:

Theorem 2.1:

If (A, p) is a complete metric space and U, V: $A \to A$ is mappings such that U is continuous, one to one and subsequentially convergent. If $\alpha \in [0, 1/2)$ and

$P(UVa,UVb) \leq \alpha\, p(Ua, UVa)+p(Ub,UVb)$, $\forall a,b \in A$, (1)

then for $\varepsilon > 0$, $T(V) \neq \phi$, that is V has approximate fixed point property.

Proof:

If a_0 be any arbitrary point in A. Define the iterative sequence $\{a_m\}$ by

$a_{m+1} = Va_m$, m=1,2,...

Now using the inequality (1) we have

$P(Ua_m , Ua_{m+1}) = p(UVa_{m-1}, UVa_m)$

$\leq \alpha\ [p(Ua_{m-1}, UVa_{m-1}) + p(Ua_m, UVa_m)]$ (2)

$P(Ua_m, Ua_{m+1}) \leq (\alpha/1-\alpha)\, p(Ua_{m-1}, Ua_m)$ (3)

By using the argument repeatedly

$$P(Ua_m, Ua_{m+1}) \leq (\alpha/1-\alpha)\, p(Ua_{m-1}, Ua_m) \leq (\alpha/1-\alpha)^2\, p(Ua_{m-2}, Ua_{m-1}) \leq ... \leq (\alpha/1-\alpha)^m\, p(Ua_0, Ua_1) \quad (4)$$

$$P(UV^m x_0, UV^{m+1} a_0) \leq (\alpha/1-\alpha)^m\, p(Ua_0, Ua_1) \quad (5)$$

Since $\alpha \in [0, 1/2)$, from the above inequality we get that $p(UV^m a_0, UV^{m+1} a_0) \to 0$ as $m \to \infty$ for all $a \in A$, which implies that V is U asymptotically regular.

Now by applying Lemma 1.6, we obtain that for every $\varepsilon > 0$,

$$T(V) \neq \phi$$

This means that V has approximate fixed point property.

Theorem 2.2:

If A be a complete b metric space with metric p and if $U: A \to A$ is a function with the following property $p(Ua, Ub) \leq x\, p(a, Ua) + y\, p(b, Ub) + z\, p(a,b)$ (6)

$\forall\, a,b \in A$, where x,y,z are non-negative real numbers and satisfying $x+v(y+z) <1$ for $v \geq 1$, then U has an approximate fixed point property.

Proof:

Let $a_0 \in A$ and $\{a_m\}$ be a sequence in A, so that

$$a_m = U a_{m-1} = U^m a_0 \quad (7)$$

Now

$$P(a_{m+1}, a_m) = p(Ua_m, Ua_{m-1})$$
$$\leq x\, p(a_m, Ua_m) + y\, p(a_{m-1}, Ua_{m-1}) + z\, p(a_m, a_{m-1})$$
$$= x\, p(a_m, a_{m+1}) + y\, p(a_{m-1}, a_m) + z\, p(a_m, a_{m-1})$$
$$\Rightarrow (1-x)\, p(a_{m+1}, a_m) \leq (y+z)\, p(a_m, a_{m-1})$$
$$\Rightarrow p(a_{m+1}, a_n) \leq ((y+z)/(1-x))\, p(a_m, a_{m-1}) = tp(a_m, a_{m-1})$$

where $t = ((y+z)/(1-x)) < 1/v$

From (7) $\Rightarrow p(U^{m+1}a_0, U^m a_0) \leq ((y+z)/(1-x))\, p(a_m, a_{m-1}) = t\, p(a_m, a_{m-1})$

Continuing the process we can easily see that $p(U^{m+1}a_0, U^m a_0) \leq t^m p(a_0, a_1)$.

Consider $n \to \infty$ then $p(U^{m+1}a_0, U^m a_0) \to 0$, for every $a \in A$.

Using the definition (3) [7] we can see U is an asymptotically regular map.

Now by applying Lemma on U we get $\varepsilon > 0$, $T(U) \neq \phi$

this means that U has an approximate fixed point property.

Theorem 2.3:

If (A, p) is a complete b-metric space with constant $v \geq 1$. Then $U : A \to A$ be a mapping so that $p(Ua, Ub) \leq h\, p(a,b)$ with $h \in [0,1)$ and $hv <1$. Then U has a spproximate fixed point property.

Proof:

Let $a_0 \in A$ and there exists a sequence $\{a_m\} \in A$ so that $a_m = U a_{m-1} = U^m a_0$, m=1,2,3,...(8)

Since U is a contraction with constant $h \in [0,1)$, then we obtain

$$P(a_{m+1}, a_m) = p(Ua_m, Ua_{m-1})$$
$$\leq h\, p(a_m, a_{m-1}) = h\, p(Ua_{m-1}, Ua_{m-2})$$
$$\leq h^2\, p(a_{m-1}, a_{m-2}) \leq ... \leq h^m\, p(a_1, a_0)$$

This implies $P(Ua_m, Ua_{m-1}) = p(U^{m+1}a_0, U^m a_0) \leq (1/v^m)\, p(a_1, a_0)$

Again if we consider $m \to \infty$ we have $p(U^{m+1}a_0, U^m a_0) \to 0$ for every $a \in A$.

By definition, U is an asymptotically regular map.

By applying the Lemma on U, we get for all $\varepsilon > 0$, $T(U) \neq \phi$

which shows that U has a fixed point property.

Conclusions:

For the maps meeting the contractive conditions Banach contraction, U contraction, Reich contraction, and Kannan contraction, we went over the approximative fixed point theorems in this article and established some new fixed point theorems for entire metric space and for b-metric space.

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