Interpolative Cirić-Reich-Rus-Type Contraction In *G*-Metric Spaces.

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Abstract
In this brief manuscript, we delve into the widely recognized Cirić·Reich-Rus-type contractions by integrating the principles of interpolation within the context of a complete G-metric space.

The investigation of fixed points in functions is a significant field of study due to its wide-ranging applications across various disciplines. The theory of fixed points is particularly valuable in understanding contractive conditions, which have practical applications in optimization, parameter estimation, and variational and linear inequalities. Consequently, fixed point theory has attracted substantial research attention. The introduction of *G* metric spaces as a generalization of metric spaces has played a crucial role in advancing fixed point theory (see [1, 2]). G-metric spaces offer a broader context for exploring fixed points of mappings and have facilitated the extension of established metric space theorems to a wider range of problems. By addressing the limitations of traditional metric spaces, G-metric spaces provide a more flexible framework for studying fixed points and have been extensively investigated and applied in diverse fields.

A *G* metric space, also known as a generalized metric space, represents an extension of the conventional metric space concept. In a G-metric space, the distance function is permitted to take values from a partially ordered set, rather than solely non-negative real numbers. This implies that the distance between two points in a *G* metric space can be in infinite or undefined, in addition to being a non-negative real number. Mustafa and Sims introduced this approach in 2004 [1], presenting it as an expansion of the traditional notion of a metric space.

Recently Karapinar [4] proposed a new Kannan-type contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called "interpolative Kannan-type contractive mapping" is a generalization of Kannan's fixed point theorem. The interpolative method has been used in other research to generalize other forms of contractions as well [4, 5]. This method has been found to be a powerful tool in the study of fixed point theory, as it allows for the construction of new classes of contractive mappings and the discovery of new fixed point theorems.

In the following sections, we will provide the necessary preliminaries and fundamental de definitions that will support the proof of our main results.

Definition 1. Let X be a nonempty set, and let $G: X \times X \times X \to \mathbb{R}^+$,a function satisfying the following properties:

1. $G(x, y, z) = 0$ if $x = y = z$
2. $G(x, y, z) > 0$, for all $x, y \in X$, with $x \neq y$
3. $G(x, x, z) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$.(Symmetry in all three variables);
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality)

Then the function G is called a generalized metric, or, more specifically a $G-$metric on X, and the pair $(X; G)$ is called a $G-$metric space.

Example 2. Let (X, d) be a usual metric space, and define G_m and G_s on $X \times X \times X \to \mathbb{R}^+$ by

$$\begin{aligned} G_m(x, y, z) &= \max\{d(x, y), d(y, z), d(x, z)\} \\ G_s(x, y, z) &= d(x, y) + d(y, z) + d(x, z) \end{aligned}$$

for all $x, y, z \in X$. Then (X, G_m) and (X, G_s) are $G-$metric spaces.

Definition 3. Let (X, G) be a $G-$metric space, and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n) if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is $G-$convergent to x.

Thus, that if $x_n \to 0$ in a $G-$mertic space (X, G) , for all $m, n \geq N$, (we mean by N the Natural numbers)

Proposition 4. Let (X, G) be a $G-$metric space. Then the following are equivalent.

1. (x_n) is $G-$convergent to x
2. $G(x_n, x_n, x) \to 0$ as $n \to \infty$
3. $G(x_n, x, x) \to 0$ as $n \to \infty$
4. $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$

Definition 5. Let (X, G) be a $G-$metric space, a sequence (x_n) is called $G-$Cauchy.

If for a given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_m, x_n, x_l) < \epsilon$, for all $n, m, l > N$

That $G(x_m, x_n, x_l) \to 0$ as $m, n, l \to \infty$

Proposition 6. In a $G-$metric space, (X, G), the following are equivalent.

1. The sequence (x_n) is $G-$Cauchy.
2. For every $\epsilon > 0$, there is $N \in \mathbb{N}$ such thaty $G(x_n, x_m, x_m) < \epsilon$, for all $n, m > N$

Definition 7. A G-metric space (X, G) is said to be G-complete (or complete $G-$metric) if every $G-$Cauchy sequence in (X, G) is G-convergent in (X, G).

In 2010 Zead Mustafai [13] proved generalized versions of Reich's results in G-metric space as follows:

Theorem 8. Let $(X; G)$ be a complete Gmetric space, and let $T: X \to X$ be a mapping satisfies the following condition: $G(Tx, Ty, Tz) \leq k[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)]$, for all $x, y, z \in X$, where $k \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point (say u), and T is $G-$continuous at u.

This theorem has been generalized in 2023 by edraoui. M, El koufi [7] using the concept of interpolation and proved a fixed point theorem in metric space

1 Main Results

The concept of interpolative Cirić-Reich-Rus type contractions extends the notions of Reich contractions and Cirić-Reich-Rus type contractions. These contractions utilize interpolation techniques, which introduce greater flexibility in their definition. This flexibility allows for the exploration of fixed points in a wider range of metric spaces and enables their application in solving various problems in optimization, approximation theory, and variational and linear inequalities. By incorporating interpolation, the theory of fixed points in metric spaces is expanded, offering a more adaptable approach for problem-solving across different fields.

In this section, we will examine an interpolative Cirić-Reich-Rus-type contraction and an interpolative Reich type contraction. Their purpose is to establish the existence of fixed points in a $G-$metric space.

Definition 9. Let (X, G) be a complete $G-$metric space, and let $T: X \to X$ is called an interpolative Reich–Rus–Cirić type contraction, if there are constants $\lambda \in [0,1)$ and $\alpha, \beta \in (0,1)$ such that:

$$G(Tx, Ty, Tz) \leq \lambda [G(x, Tx, Tx)]^{\alpha} \cdot [G(y, Ty, Ty)]^{\beta} \cdot [G(z, Tz, Tz)]^{1-\alpha-\beta} \qquad (2)$$

for all $x, y, z \in X \setminus Fix(T)$

Theorem 10. Let (X, G) be a complete $G-$metric space, and T be an interpolative Reich–Rus–Cirić type contraction. Then T has a unique fixed point in X.

Proof. Given an arbitrary point $x_0 \in X$, construct a sequence x_n by $x_n = T^n(x_0)$, then by ([1]) we have

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &= G\big(T^n(x_0), T^{n+1}(x_0), T^{n+1}(x_0)\big) \\
&\le \lambda\big[G\big(T^{n-1}(x_0), T^n(x_0), T^n(x_0)\big)\big]^{\alpha} \cdot \big[G\big(T^n(x_0), T^{n+1}(x_0), T^{n+1}(x_0)\big)\big]^{\beta} \cdot \big[G\big(T^n(x_0), T^{n+1}(x_0), T^{n+1}(x_0)\big)\big]^{1-\alpha-\beta} \\
&\le \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha} \cdot [G(x_n x_{n+1}, x_{n+1})]^{\beta} \cdot [G(x_n, x_{n+1}, x_{n+1})]^{1-\alpha-\beta} \\
&\le \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha} \cdot [G(x_n, x_{n+1}, x_{n+1})]^{1-\alpha}
\end{aligned}$$

which yields that

$$[G(x_n, x_{n+1}, x_{n+1})]^{1-\alpha} \le \lambda[G(x_{n-1}, x_n, x_n)]^{1-\alpha} \qquad (3)$$

And so

$$G(x_n, x_{n+1}, x_{n+1}) \le kG(x_{n-1}, x_n, x_n) \text{ where } k = \lambda^{\frac{1}{1-\alpha}} \quad \text{and clearly } k \in (0,1).$$

Thus, we have:

$$G(x_n, x_{n+1}, x_{n+1}) \le k^n G(x_0, x_1, x_1) \qquad (4)$$

Moreover, for all $n, m \in \mathbb{N}$, such that $n < m$

Now,we will demonstrate that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$$

By employing the rectangle inequality multiple times and utilizing equation (4), we can deduce.

$$\begin{aligned}
G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + G(x_{m-1}, x_m, x_m) \\
&\le (k^n + k^{n+1} + \ldots + k^{m-1}) G(x_0, x_1, x_1) \\
&\le \frac{k^n}{1-k} G(x_0, x_1, x_1)
\end{aligned}$$

Then $G(x_n, x_m, x_m) \to 0$ as $m, n \to \infty$, , thus (x_n) is G-Cauchy sequence. then by completeness of (X, G), there exists $z \in X$ such that (x_n) is G-convergent to z. Suppose that $T(z) \neq z$. Then.

$$\begin{aligned}
G(x_n, Tz, Tz) &\le \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha} \cdot [G(z, Tz, Tz)]^{\beta} \cdot [G(z, Tz, Tz)]^{1-\alpha-\beta} \\
&\le \lambda[G(x_{n-1}, x_n, x_n)]^{\alpha} \cdot [G(z, Tz, Tz)]^{1-\alpha}
\end{aligned}$$

Taking the limit as $n \to \infty$ and using the fact that the function G is continuous on its variables, we have $G(z, Tz, Tz) = 0$ which is a contradiction since $T(z) \neq z$.To establish uniqueness, suppose that $y \neq z$ is such that $Ty = y$ then (4) implies that:

$$\begin{aligned}
G(y, z, z) &= G(Ty, Tz, Tz) \le \lambda[G(y, Ty, Ty)]^{\alpha} \cdot [G(z, Tz, Tz)]^{\beta} \cdot [G(z, Tz, Tz)]^{1-\alpha-\beta} \\
&\le \lambda[G(y, Ty, Ty)]^{\alpha} \cdot [G(z, Tz, Tz)]^{1-\alpha} \\
&\le \lambda[G(y, y, y)]^{\alpha} \cdot [G(z, z, z)]^{1-\alpha} = 0
\end{aligned}$$

which yields that $G(y, z, z) = 0$ a contradiction. Hence, the observed fixed point is unique.

Corollary 11 . Let (X, G) be a complete $G-$metric space, and let $T: X \to X$ be a mapping satisfying the following condition for some $m \in \mathbb{N}$,

$$G(T^m x, T^m y, T^m z) \le \lambda[G(x, T^m x, T^m x)]^{\alpha} \cdot [G(y, T^m y, T^m y)]^{\beta} \cdot [G(z, T^m z, T^m z)]^{1-\alpha-\beta} \qquad (5)$$

for all $x, y, z \in X \setminus \mathrm{Fix}(T)$ and $\lambda \in [0,1)$ and $\alpha, \beta \in (0,1)$. Then T has unique fixed point.

Proof. Let's put $S = T^m$. Using condition (5) then S satisfies

$$G(Sx, Sy, Sz) \le \lambda[G(x, Sx, Sx)]^{\alpha} \cdot [G(y, Sy, Sy)]^{\beta} \cdot [G(z, Sz, Sz)]^{1-\alpha-\beta}$$

That is S is on interpolative Reich–Rus–Cirić type contraction. Thus, by Theorem 10, S has a unique fixed point x, that is $Sx = x$. This proves that x is unique fixed point of T^m

Then $x = T^m(x)$. But $T(x) = T(T^m(x)) = T^{m+1}(x) = T^m\big(T(x)\big)$, so $T(x)$ is another fixed point for T^m and by uniqueness $T(x) = x$

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