Bounds On The Probability Of Error In Term Of Generalized Renyi's Entropy

R. K. Dahiya[1]*

[1]*HOD, Department of Mathematics Pt. NRS Govt. College, Rohtak –Haryana and Hardik Dahiya Vellore Institute of Technology, Chennai

***Corresponding Authors:** R. K. Dahiya
*HOD, Department of Mathematics Pt. NRS Govt. College, Rohtak –Haryana and Hardik Dahiya Vellore Institute of Technology, Chennai

Abstract:
Bounds on the probability of error in Term of generalized Renyi's for two class cases are obtained. These bounds are valid for $0 < \alpha < 1$ and $\alpha \geq 2$. The obtained result had more significant, because Renyi's entropy gave better result for higher order, i.e. for $\alpha > 1$.

1. Introduction

Consider two finite probability distribution p = (p_1, p_2, p_3, …….., p_n),

$$p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1\ and\ Q = (q_1, q_2, q_3, ..., q_n),\ q_i \geq 0,\ \sum_{i=1}^{n} q_i = 1$$

Shannon's Entropy (1948) and its generalization, I.e. Inaccuracy is given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \qquad ....(1.1.1)$$

$$H(P//Q) = -\sum_{i=1}^{n} p_i \log q_i \qquad ....(1.1.2)$$

In (1.1.1), and (1.1.2), if H(P) and H(P//Q) is replaced Entropy (1961), i.e.,

$$R_\alpha(P) = \left(\frac{1}{1-\alpha}\right) \log \sum_{i=1}^{n} p_i^\alpha,\ \alpha \neq 1,\ \alpha > 0 \qquad ....(1.1.3)$$

$$R_\alpha(P//Q) = \left(\frac{1}{1-\alpha}\right) \log \sum_{i=1}^{n} p_i^\alpha\, q_i^{1-\alpha},\ \alpha \neq 1,\ \alpha > 0 \qquad ......(1.1.4)$$

From (1.1.3),

$$(1-\alpha) R_\alpha(P) = \log \sum_{i=1}^{n} p_i^\alpha$$

$$2^{(1-\alpha) R_\alpha(P)} = \sum_{i=1}^{n} p_i^\alpha \qquad ......(1.1.5)$$

If we replace P by Q in above, then

$$2^{(1-\alpha) R_\alpha(Q)} = \sum_{i=1}^{n} q_i^\alpha \qquad .....(1.1.6)$$

Adding (1.1.5) and (1.1.6), we get,

$$\frac{1}{2}\left[2^{(1-\alpha)R_\alpha(p)} + 2^{(1-\alpha)R_\alpha(q)}\right] = \frac{1}{2}\left[\sum_{i=1}^{n} p_i^\alpha + \sum_{i=1}^{n} q_i^\alpha\right]$$

$$\left(\frac{1}{1-\alpha}\right) \log\left[\frac{1}{2}\left\{2^{(1-\alpha)R_\alpha(P)} + 2^{(1-\alpha)R_\alpha(Q)}\right\}\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\frac{1}{2}\left\{\sum_{i=1}^{n}p_i^{\alpha}+\sum_{i=1}^{n}q_i^{\alpha}\right\}\right]$$

Let,

$$M_{\alpha}^{1}(P//Q)=\left(\frac{1}{1-a}\right)\log\left[\frac{1}{2}\left\{2^{(1-\alpha)R_{\alpha}(P)}+2^{(1-\alpha)R_{\alpha}(Q)}\right\}\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\frac{1}{2}\left\{\sum_{i=1}^{n}p_i^{\alpha}+\sum_{i=1}^{n}q_i^{\alpha}\right\}\right] \qquad .....(1.1.7)$$

From (1.1.4),

$$(1-\alpha)R_{\alpha}(P//Q)=\log\sum_{i=1}^{n}\left(p_i^{\alpha}\,q_i^{1-\alpha}\right)$$

$$2^{(1-\alpha)R_{\alpha}(P//Q)}=\sum_{i=1}^{n}\left(p_i^{\alpha}\,q_i^{1-\alpha}\right)$$

$$2^{(1-\alpha)R_{\alpha}\left(P//\frac{P+Q}{2}\right)}=\sum_{i=1}^{n}\left(p_i^{\alpha}\left\{\frac{1}{2}(p_i+q_i\right\}^{1-\alpha}\right) \qquad .....(1.1.8)$$

Similarly we have, $2^{(1-\alpha)R_{\alpha}\left(Q//\frac{P+Q}{2}\right)}=\sum_{i=1}^{n}\left(q_i^{\alpha}\left\{\frac{1}{2}(p_i+q_i)\right\}^{1-\alpha}\right)$ .....(1.1.9)

Adding (1.1.8) and (1.1.9), we get,

$$\frac{1}{2}\left[2^{(1-\alpha)R_{\alpha}\left(P//\frac{P+Q}{2}\right)+2^{(1-\alpha)}R_{\alpha}\left(Q//\frac{P+Q}{2}\right)}\right]$$

$$=\frac{1}{2}\left[\sum_{i=1}^{n}\left(p_i^{\alpha}\left\{\frac{1}{2}(p_i+q_i)\right\}^{1+\alpha}\right)+\sum_{i=1}^{n}\left(q_i^{\alpha}\left\{\frac{1}{2}(p_i+q_i\right\}^{1-\alpha}\right)\right]$$

$$=\sum_{i=1}^{n}\left(\left\{\frac{p_1^{\alpha}+q_i^{\alpha}}{2}\right\}\left\{\frac{p_i+q_i}{2}\right\}^{1-\alpha}\right)$$

$$\left(\frac{1}{1-\alpha}\right)\log\left(\frac{1}{2}\left[2^{(1-\alpha)R_{\alpha}\left(P//\frac{P+Q}{2}\right)}+2^{(1-\alpha)R_{\alpha}\left(Q//\frac{P+Q}{2}\right)}\right]\right)$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left(\sum_{i=1}^{n}\left(\left\{\frac{p_i^{\alpha}+q_i^{\alpha}}{2}\right\}\left\{\frac{p_i+q_i}{2}\right\}^{1-\alpha}\right)\right)$$

Let,

$$M_{\alpha}^{2}(P//Q)=\left(\frac{1}{1-\alpha}\right)\log\left(\frac{1}{2}\left[2^{(1-\alpha)R_{\alpha}\left(P//\frac{P+Q}{2}\right)}+2^{(1-\alpha)R_{\alpha}\left(Q//\frac{P+Q}{2}\right)}\right]\right)$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left(\sum_{i=1}^{n}\left(\left\{\frac{p_i^{\alpha}+q_i^{\alpha}}{2}\right\}\left\{\frac{p_i+q_i}{2}\right\}^{1-\alpha}\right)\right);\ \alpha\neq 1,\alpha>1 \qquad .....(1.1.10)$$

Since, logarithmic function is positive, hence, $R_{\alpha}(P)$ also positive. So both $M_{\alpha}^{1}(P//Q)$ and $M_{\alpha}^{2}(P//Q)$, for all $\alpha \neq 1$, $\alpha > 0$ are guaranteed non negative . The aim of this chapter is to present probability bounds on the error by taking equation (1.1.7) and (1.1.10) into consideration.

1.2 Error Probability for case k_1 and k_2

Let we have a priori probability $p_1 = p(k_1)$, i = 1, 2 with two case k_1 and k_2 . Let a class conditional density function $p(z/k_1)$ have the feature z on Z. we calculate the conditional probability p(K/z) of $K = k_i$, i = 1, 2, By Bayes rule. The minimum possible probability of error is given by

$$P_e = \int \min_i \{p(z/k_i) p_i\} dz,\ i = 1,2 = \int P_e(z) p(z) dz\ ,$$

where,

$$P_e(z) = \min\{p(k_1/z_1), p(k_2/z)\},\ p(z) = p_1 p(z/k_1) + p_2 p(z/k_2),$$
$$p_1 p(z/k_1) = p(z) p(k_1/z),\ and\ \ p_2 p(z/k_2) = p(z) p(k_2/z),$$

Generalization of $M_\alpha^1(P//Q)\ and\ M_\alpha^2(P//Q)$, in term of the prior probability for the two class case, is given by,

$$M_\alpha^1 = \begin{cases} \dfrac{1}{1-\alpha}\log\left\{\int p(z/k_1)^\alpha\ dz + \int p(z/k_2)^\alpha\ dz\right\} & ;\alpha \neq 1,\ \alpha > 0 \\ \dfrac{1}{2}\left(\left\{-\int p\left(\dfrac{z}{k_1}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_1}\right)dz\right\} - \left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\}\right) & ;\alpha = 1 \end{cases}$$

$$M_\alpha^2 = \begin{cases} \left(\dfrac{1}{1-\alpha}\right)\log\left[\int \left\{\dfrac{p\left(\dfrac{z}{k_1}\right)^\alpha + p\left(\dfrac{z}{k_2}\right)^\alpha}{2}\right\}\left\{\dfrac{p\left(\dfrac{z}{k_1}\right) + p\left(\dfrac{z}{k_2}\right)}{2}\right\}^{1-\alpha} dz\right] & ;\alpha \neq 1, \alpha = 0 \\ \dfrac{1}{2}\left(\left\{-\int p\left(\dfrac{z}{k_1}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_1}\right)dz\right\} - \left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\}\right) & ;\alpha = 1 \end{cases}$$

More general way of, defining these measure is as follows:

$$M_\alpha^1(p_1, p_2) = \begin{cases} \left(\dfrac{1}{1-\alpha}\right)\log\left\{\dfrac{\int p_1 p_2\left(\dfrac{z}{k_1}\right)^\alpha dz + \int p_2 p\left(\dfrac{z}{k_2}\right)^\alpha dz}{2}\right\} & ;\alpha \neq 1, \alpha > 0 \\ \dfrac{1}{2}\left\{-\int p\left(\dfrac{z}{k_1}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_1}\right)dz\right\} - \left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\} & ;\alpha = 1 \end{cases} \quad ...(1.2.1)$$

$$M_\alpha^2(p_1, p_2) = \begin{cases} \left(\dfrac{1}{1-\alpha}\right)\log\left[\int\left\{\dfrac{\left((p_1 p(z/k_1)\right)^\alpha + \left(p_2 p(z/k_2)\right)^\alpha}{2} \times \left\{\dfrac{p_1 p(z/k_2) + p_2 p(z/k_2)}{2}\right\}^{1-\alpha} dz\right\}\right] \\ \dfrac{1}{2}\left(\left\{-\int p\left(\dfrac{z}{k_1}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_1}\right)dz\right\} - \left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\}\log\left\{\int p\left(\dfrac{z}{k_2}\right)dz\right\}\right); & \alpha = 1 \end{cases} \quad ....(1.2.2)$$

Now putting $P_1 = P_2 = \dfrac{1}{2}$ in equation (1.2.1) and (1.2.2)

$$M_\alpha^1\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{1-\alpha}\right)\log\left[\frac{\int\left(\dfrac{p(z/k_1)}{2}\right)^\alpha dz + \int\left(\dfrac{p(z/k_2)}{2}\right)^\alpha dz}{2}\right];\ \alpha \neq 1,\ \alpha > 0$$

$$= \left(\frac{1}{1-\alpha}\right)\log\left[\frac{\left(\dfrac{1}{2}\right)^\alpha\left\{\int\left(p\left(\dfrac{z}{k_1}\right)\right)^\alpha dz + \int\left(p\left(\dfrac{z}{k_2}\right)\right)^\alpha dz\right\}}{2}\right];\ \alpha \neq 1,\ \alpha > 0$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\frac{\left\{\int\left(p\left(\frac{z}{k_1}\right)\right)^{\alpha}dz+\int\left(p\left(\frac{z}{k_2}\right)\right)^{\alpha}dz\right\}}{2}\right]+\left(\frac{1}{1-\alpha}\right)\log\left(\frac{1}{2}\right)^{\alpha}$$

$$=M_{\alpha}^{1}+\left(\frac{\alpha}{1-\alpha}\right);\ \alpha\neq 1,\ \alpha>0 \quad ....(1.2.3)$$

$$M_{\alpha}^{2}\left(\frac{1}{2},\frac{1}{2}\right)=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left\{\left(\frac{p(z/z_1)}{2}\right)^{\alpha}+\left(\frac{p(z/k_2)}{2}\right)^{\alpha}\right\}\times\left\{\frac{\frac{p(z/k_1)}{2}+\frac{p(z/k_2)}{2}}{2}\right\}^{1-\alpha}dz\right]\alpha\neq 1,\ \alpha>0$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left\{\frac{\left(\frac{1}{2}\right)^{\alpha}\left((p(z/k_1))^{\alpha}+\left((p(z/k_2)^{\alpha}\right)\right)}{2}\right\}\times\left(\frac{1}{2}\right)^{1-\alpha}\left\{\frac{p(z/k_1)+p(z/k_2)}{2}\right\}^{1-\alpha}dz\right];\alpha\neq 1,\alpha>0$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left\{\frac{\left((p(z/k_1))^{\alpha}+\left((p(z/k_2)^{\alpha}\right)\right)}{2}\right\}\left\{\frac{p(z/k_1)+p(z/k_2)}{2}\right\}^{1-\alpha}dz\right]$$

$$+\left(\frac{1}{1-\alpha}\right)\log\left(\frac{1}{2}\right)=M_{\alpha}^{2}+\left(\frac{1}{1-\alpha}\right);\alpha\neq 1,\alpha>0 \quad ...(1.2.4)$$

We can write, by Bayes rule,

$$M_{\alpha}^{2}(p_1,p_2)=\left(\frac{1}{1-\alpha}\right)\log\left[\int M_{\alpha}^{1}(z)p^{\alpha}(z)dz\right];\ \alpha>0 \quad ...(1.2.5)$$

$$M_{\alpha}^{2}(p_1,p_2)=\left(\frac{1}{1-\alpha}\right)\log\left[\int M_{\alpha}^{2}(z)p(z)dz\right];\ \alpha>0 \quad ...(1.2.6)$$

Where,

$$M_{\alpha}^{1}(z)=2^{(1-\alpha)H_{\alpha}\left(\frac{K}{(Z=z}\right)}$$

$$M_{\alpha}^{2}(z)=2^{(1-\alpha)H_{\alpha}\left(\frac{K}{(Z=z}\right)2^{(1-\alpha)}} \quad and$$

$$H_{\alpha}(K/Z=z)=\begin{cases}\left(\frac{1}{1-\alpha}\right)\log\left[p^{\alpha}\left(\frac{k_1}{z}\right)+p^{\alpha}\left(\frac{k_2}{z}\right)\right] & ;\alpha\neq 1,\ \alpha>0\\ -\left(p\left(\frac{k_1}{z}\right)\right)\log\left(p\left(\frac{k_1}{z}\right)\right)-\left(p\left(\frac{k_2}{z}\right)\right)\log\left(p\left(\frac{k_2}{z}\right)\right) & ;\alpha=1\end{cases}$$

From equation (1.2.5)

$$M_{\alpha}^{1}(p_1p_2)=\left(\frac{1}{1-\alpha}\right)\log\left[\int M_{\alpha}^{1}(z)p^{\alpha}(z)dz\right] \quad =\left(\frac{1}{1-\alpha}\right)\log\left[\int 2^{(1-\alpha)H_{\alpha}9K//(Z=z))}p^{\alpha}(z)dz\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int 2^{(1-\alpha)\left(\frac{1}{1-\alpha}\right)\log\left[p^{\alpha}(k_1+z)+p^{\alpha}(k_2/z)\right]}p^{\alpha}(z)dz\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left(p^{\alpha}(k_1/z)+p^{\alpha}(k_2/z)\right)p^{\alpha}(z)dz\right] \quad ....(1.2.8)$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left(p_1p^{\alpha}(z/k_1)+p_2p^{\alpha}(z/k_2)\right)dz\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int M_\alpha^1(z)p^\alpha(z)dz\right];\ \alpha>0$$, Similarly from equation, (1.2.6)

$$M_\alpha^2(p_1,p_2)=\left(\frac{1}{1-\alpha}\right)\log\left[\int M_\alpha^2(z)p(z)dz\right] \qquad =\left(\frac{1}{1-\alpha}\right)\log\left[2^{(1-\alpha)H^\alpha(K//(Z=z))2^{1-\alpha}}\ p(z)dz\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int 2^{(1-\alpha)\left(\left(\frac{1}{1-\alpha}\right)\log\left[p^\alpha(k_1/z)+p^\alpha(k_2/z\right]\right)2^{1-\alpha}}\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left[p^\alpha\left(\frac{k_1}{z}\right)+p^\alpha\left(\frac{k_2}{z}\right)\right]2^{(\alpha-2)}p(z)dz\right] \qquad \ldots (1.2.9)$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left(\frac{p^\alpha(z)p^\alpha(k_1/z)+p^\alpha(z)p^\alpha(k_2/z)}{2}\right)\left(\frac{p(z)}{2}\right)^{1-\alpha}dz\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left(\frac{p_1p^\alpha(z/k_1)+p_2p^\alpha(z/k_2)}{2}\times\frac{p_1p(z/k_1)+p_2p(z/k_2)}{2}\right)^{1-\alpha}dz\right]$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left(\int M_\alpha^2(z)\,p(z)\,dz\right)$$

From (1.2.8),

$$M_\alpha^1(p_1,p_2)=\left(\frac{1}{1-\alpha}\right)\log\left[\int\left(p^\alpha\left(\frac{k_1}{z}\right)+p^\alpha\left(\frac{k_2}{z}\right)\right)p^\alpha(z)\,dz=H_\alpha\left(\frac{K}{Z}\right)\right] \ldots (1.2.10)$$

From (1.2.9),

$$M_\alpha^2(p_1,p_2)=\left(\frac{1}{1-\alpha}\right)\log\left(\int\left\{p^\alpha\left(\frac{k_1}{z}\right)+p^\alpha\left(\frac{k_2}{z}\right)\right\}2^{\alpha-2}\ p(z)\,dz\right)$$

$$=\left(\frac{1}{1-\alpha}\right)\log\left(\int\left\{p^\alpha\left(\frac{k_1}{z}\right)+p^\alpha\left(\frac{k_2}{z}\right)\right\}p(z)\,dz\right)+\left(\frac{1}{1-\alpha}\right)\log\ 2^{\alpha-2}$$

$$=H_\alpha\left(\frac{K}{Z}\right)+\left(\frac{\alpha-2}{1-\alpha}\right)=H_\alpha\left(\frac{K}{Z}\right)+\left(\frac{1}{1-\alpha}\right)-1;\ \alpha>0 \quad \ldots. (1.2.11)$$

But Bassat (1978), Devizver (1977), and Taneja (1983) defines,

$$2P_e\le H_\alpha\left(\frac{K}{Z}\right)\le H_\alpha(P_e,1-P_e) \qquad \ldots. (1.2.12)$$

From equation (1.2.10) and (1.2.11) $M_\alpha^1(p_1,p_2)=\begin{cases}\ge M_\alpha^2(p_1,p_2) & ;0<\alpha\le 1,\ \alpha\ge 2\\ \ge M_\alpha^2(p_1,p_2) & ;1<\alpha\le 2\end{cases}$

The expressions (1.2.5) – (1.2.12) gives the following bounds:

(a) In Posterior probabilities terms:

(1) $$P_e\le\left(\frac{1-\left(\frac{1}{1-\alpha}\right)+M_\alpha^2(p_1,\ p_2)}{2}\right)\le\left(\frac{1-\left(\frac{1}{1-\alpha}\right)+M_\alpha^1(p_1,\ p_2)}{2}\right);\ 0<\alpha\le 1,\ \alpha\le 2$$

(2) $$(1-H_\alpha\{P_e,1-p_e\})\le\left(\left(\frac{1}{1-\alpha}\right)-M_\alpha^2(p_1,p_2)\right)\le\left(\left(\frac{1}{1-\alpha}\right)-M_\alpha^1(p_1,p_2)\right);\ 1<\alpha\le 2$$

(b) In prior probabilities terms:

(1) $$P_e\le\left(\frac{1+M_\alpha^2}{2}\right)\le\left(\frac{2+M_\alpha^1}{2}\right);\ 0<\alpha\le 1,\ \alpha\ge 2$$

(2) $$(1-H_\alpha\{P_e,1-P\})\le(-M_\alpha^2)\le(-\{1+M_\alpha^1\});1<\alpha\le 2$$

3. Conclusions

Thus, bounds for error probability have been obtained for two class case, and valid for 0 < α ≤ 1 and α ≥ 2. The inequality $M_{\alpha}^{1}(p_1,\ p_2)$ *and* $M_{\alpha}^{2}(p_1,\ p_2)$ have negative sign for the value 1 < α ≤ 2. The obtained result had more significant, because Renyi's Entropy for higher order, i.e. for α > 1, gives better result.

4. References

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