$\text{Generalized}_{\mathcal{F}} - \text{Topology}$

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ABSTRACT. In this paper we defined and characterized the concept of generalized fuzzy topological space($\text{generalized}_{\mathcal{F}} - \text{topological space}$) and obtained some significant results in this context with help of various supporting examples.

Keywords: Fuzzy open set, fuzzy topological space, $\text{generalized}_{\mathcal{F}} - \text{topological space}$

1. Introduction

One of the earliest branches of mathematics which applied fuzzy set theory systematically is General Topology. Although fuzzy topology is a generalization of topology in classical mathematics, it has its own marked characteristics. Azad [1] has introduced the concept of fuzzy semi-open sets in fuzzy topological spaces. Chang, C.L introduced the concept of fuzzy topological spaces [2]. Csaszar [3] introduced the notions of generalized topological spaces. Palani Cheety [4] introduced the concept of generalized fuzzy topology and investigates various properties.
In this paper, we have introduced the concept of $\text{generalized}_{\mathcal{F}} -$topological space and verify the results with the help of some counter examples. Some require basic definitions, concepts of topological space and notations are discussed in Section 2. The section 3 has been headed by $\text{generalized}_{\mathcal{F}} - \text{topological space}$, in which we verified various results related it by giving suitable examples and the Section 4 concludes the paper.

2. Preliminaries

Definition 2.1: Let X be a non-empty universal crisp set. A **fuzzy topology** on X is a non-empty collection τ of fuzzy sets on X satisfying the following conditions
i) Fuzzy sets 0 and 1 belong to τ
ii) Any arbitrary union of members of τ is in τ
iii) A finite intersection of members of τ is in τ

Here 0 and 1 represent the Zero Fuzzy Set and the Whole Fuzzy set on X, defined as, 0(x)=0,$\forall$ x $\in$ X 1(x)=1 ,$\forall\, x \in X$ and the pair (X, τ) is called Fuzzy Topological Space on X. For Convenience, we shall denote the fuzzy topological space simply as X.

Definition 2.2: Let (X, τ)be fuzzy topological space. The members of the collection τ are called **fuzzy open sets** of fuzzy topological space X. The complement of a fuzzy open set of X is called a **fuzzy closed set**. Thus, a fuzzy set λ on X is a fuzzy closed set in (X, τ) if its complement λ^C is fuzzy open set in X with respect to fuzzy topology τ.

Definition 2.3: Let (X, τ) be a fuzzy topological space. For a fuzzy set A in X the **closure** of A, denoted by Cl (A) is defined as $\text{Cl}(A) = \inf\{K: A \subseteq K, K^C \in \tau\}$. Thus the fuzzy set Cl (A) is the smallest fuzzy closed set in X containing the fuzzy set A. From the definition, if follows that Cl (A) is the intersection of all fuzzy closed sets in X containing A.

Definition 2.4: Let (X, τ) be a fuzzy topological space. For a fuzzy set A in X, the **interior** of A, denoted by Int(A) is defined as $\text{Int}(A) = \text{Sup}\{Q : Q \subseteq A, Q \in \tau\}$. Thus the fuzzy set Int (A) is the largest fuzzy open set in X contained in the fuzzy set A. From definition, it follows that Int(A) is the union of all fuzzy open sets in X contained in A.

Proposition 2.1: Let (X, τ) be a fuzzy topological space. Then
i) Arbitrary Intersection of fuzzy closed sets is a fuzzy closed set.
ii) Finite union of fuzzy closed sets is a fuzzy closed set.

Proposition 2.2: Let(X, τ) be a fuzzy topological space and let A be a fuzzy set in X. Then
i) $\text{Cl}(A) = A$ if and only if A is a fuzzy closed set in X.
ii) $\text{Int}(A) = A$ if and only if A is a fuzzy open set in X.

3. $\textbf{Generalized}_{\mathcal{F}} - \textbf{topological}\ \mathcal{S}\textbf{pace}$

Definition 3.1: Let X be a crisp set and let μ be a collection of fuzzy sets on X. Then μ is called $\text{generalized}_{\mathcal{F}} - \text{topology}$on X if it satisfies following conditions

i) The fuzzy sets 0 and 1 are in μ where $0,1: X \to I$ are defined as $0(x) = 0$ and $1(x) = 1$ for all $x \in X$

ii) If $\{\lambda_j\}$, $j \in J$ is any family of fuzzy sets on X where $\lambda_j \in \mu$ then $\cup_{j\in J} \lambda_j \in \mu$

The pair (X, μ) is called $\text{generalized}_{\mathcal{F}} - \text{topological}\ \mathcal{S}\text{pace}$

Definition 3.2: Let (X, μ) be $\text{generalized}_{\mathcal{F}} - \text{topological}\ \mathcal{S}\text{pace}$. The members of the collection μ are called $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ in$\text{generalized}_{\mathcal{F}} - \text{topological}\ \mathcal{S}\text{pace}$. The complement of $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ in X is called $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{lose}\ \mathcal{S}\text{et}$

Example 3.1: Let $X = \{x_1, x_2\}$, and we consider fuzzy sets $A = \{(x_1, 0.3), (x_2, 0.6)\}$, $B = \{(x_1, 0.5), (x_2, 0.4)\}$ and $C = \{(x_1, 0.5), (x_2, 0.6)\}$ on X. Then clearly $\mu = \{0, A, B, C, 1\}$ is $\text{generalized}_{\mathcal{F}} - \text{topology}$ on X, but not fuzzy topology on X.

Definition 3.3: Let (X, μ) be $\text{generalized}_{\mathcal{F}} - \text{topological}\ \mathcal{S}\text{pace}$. For a fuzzy set A in X the $\mathcal{C}$losure of A is defined as $Cl_\mu(A) = \inf\{K : A \subseteq K, K^C \in \mu\}$. Thus $Cl_\mu(A)$ is the smallest $\mathcal{C}\text{losed}\ \mathcal{S}\text{et}$ in X containing the fuzzy $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ A. From the definition, if follows that $Cl_\mu(A)$ is the intersection of all $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{ets}$ in X containing A.

Definition 3.4: Let (X, μ) be $\text{generalized}_{\mathcal{F}} - \text{topological}\ \mathcal{S}\text{pace}$. For a fuzzy $\mathcal{S}$et A in X, the $\mathcal{I}$nterior of A, is defined as $I_\mu(A) = \text{Sup}\{Q : Q \subseteq A, Q \in \mu\}$. Thus $I_\mu(A)$ is the largest $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ in X contained in the fuzzy $\mathcal{S}$et A. From the definition, if follows that $I_\mu(A)$ is the union of all $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ in X contained in A.

Remark 3.1: Arbitrary union of $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ is $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$

Proposition 3.1: Let (X, μ) be $\text{generalized}_{\mathcal{F}} - \text{topological}\ \mathcal{S}\text{pace}$.Then:

i) 0 and 1 are fuzzy $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{ets}$ in X.

ii) Arbitrary intersection of $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{ets}$ in X is $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{et}$ in X.

Proof (i): Since 0 and 1 are $\text{generalized}_{\mathcal{F}} -$ open $\mathcal{S}ets$ in X therefore their complement 1 and 0 are $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{etS}$ in X.

(ii): Let $\{\lambda_j\}_{j\in J}$ be a collection of $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{ets}$ in X, where J is index set. Then $\{\lambda_j^c\}_{j\in J}$ is a collection of $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{ets}$ in X. This implies $\cup_{j\in J} \lambda_j^c$ is $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ in X. Hence $(\cup_{j\in J}\ \lambda_j^c)^c = \cap_{j\in J} \lambda_j$ is $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{et}$ in X.

Remark 3.2: Since arbitrary union of $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ is $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$, $I_\mu(\lambda)$ is $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ in X. Further since arbitrary intersection of $\text{generalized}_{\mathcal{F}} - \text{closed}\ \mathcal{S}\text{et}$ is fuzzy closed $\mathcal{S}$et ., $Cl_\mu(\lambda)$ is a $\text{generalized}_{\mathcal{F}} - \text{closed}\ \mathcal{S}\text{et}$ in X. Intersection of two $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{ets}$ may not $\text{generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ and therefore union of two $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{ets}$ may not $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{et}$ in X. In Example 2..1 we see that $A \cap B = \{(x_1, 0.3), (x_2, 0.4)$ is not $\text{Generalized}_{\mathcal{F}} - \mathcal{O}\text{pen}\ \mathcal{S}\text{et}$ in X and $A \cup B = \{(x_1, 0.7), (x_2, 0.6)\}$ is not $\text{generalized}_{\mathcal{F}} - \mathcal{C}\text{losed}\ \mathcal{S}\text{et}$ in X.

Proposition 3.2: let $\{\mu_j\}_{j\in J}$be a collection of $\text{generalized}_{\mathcal{F}} - \text{topologies}$ on X. where J is an index set then their intersection $\cap_{j\in J}\ \mu_j$ is also a $\text{generalized}_{\mathcal{F}} - \text{topology}$ on X

Proof: let $\{\mu_j\}_{j\in J}$ be a collection of $\text{generalized}_{\mathcal{F}} - \text{topologies}$ on X.where J is an arbitrary index set be a collection of $\text{generalized}_{\mathcal{F}} - \mathcal{T}\text{opologies}$ on X. Then $0,1 \in \{\mu_j\}_{j\in J}$ for all $j \in J$. This implies $0,1 \in \cap_{j\in J} \{\mu_j\}$. Now let $A_\alpha \in \cap_{j\in J} \{\mu_j\}$ for $\alpha \in J_1$ where J_1 is an arbitrary index set. Then $A_\alpha \in \{\mu_j\}_{j\in J}$ for all $j \in J$ and for all $\alpha \in .\ J_1$. Since each $\{\mu_j\}_{j\in J}$ be a collection of $\text{generalized}_{\mathcal{F}} - \mathcal{T}\text{opologies}$ on X. it follow that $\cup_{\alpha\in J_1} A_\alpha \in \mu_j$ for all $j \in J$. Hence $\cup_{\alpha\in J_1} A_\alpha \in \cap_{j\in J} \mu_j$. Thus $\cap_{j\in J} \{\mu_j\}$.is $\text{generalized}_{\mathcal{F}} - \text{topology}$ on X. However collection of $\text{generalized}_{\mathcal{F}} - \mathcal{T}\text{opology}$ on X is not closed set under the operation of union i.e. union of two $\text{generalized}_{\mathcal{F}} - \text{topologies}$ is not necessarily $\text{generalized}_{\mathcal{F}} - \text{topology}$.

4. Conclusion

In this Paper we have studied a new concept of generalized fuzzy topological spaces in which many important results have been obtained with the help of some suitable examples.

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