Analysis of 2-D Steady State Heat Conduction In A Slab Subjected With Different Types of Boundary Conditions Using Method of Lines

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Abstract:
Many disciplines in science and engineering, heat transfer plays a vital role and the problems associated with heat transfer are of great importance. Generally, two-dimensional steady state conduction is governed and transformed into a second order partial differential equation (PDE). Satisfying the differential equation along with four boundary conditions is essential for a solution to be valid. There are analytical solutions available, but only for simple boundary conditions and these are not suitable for complex boundary conditions. A technique for solving partial differential equations is Method of Lines (MOL), in which one dimension is discretized. In this study an analytical approach to a two-dimensional slab with steady state heat conduction under different types of boundary conditions is considered. The complete description of heat flow through slab using MOL is presented and the obtained PDEs are solved. Temperature distribution profiles in the slab by using MOL were plotted and compared with the profiles of analytical solutions by using MATLAB. The steady state analysis of temperature distribution in a slab with specified Dirichlet boundary conditions and Neumann boundary conditions was developed by using Method of Lines. From the profiles it is observed that as the number of lines increases, the error between analytical and MOL (semi analytical) solutions decreases, and the profiles of analytical and MOL converged which indicates the preference of higher number of lines in order to obtain accurate values.

Keywords: Heat transfer and conduction, Analytical approach, Semi-Analytical method, differential equations, steady state.

1. Introduction:

Most of the physical world problems are described in terms of science and engineering with respect to the analysis of unsteady, heat conduction, one dimensional, 2-dimensional and 3dimensional space and time. Out of the most popular forms of mathematics, PDEs fall into special category since they provide the description of physical space time in mathematical form. For the engineering systems to be modelled, especially chemical engineering systems PDEs are of wide interest. PDE models are generally obtained from the basics of mass balance, energy balance and momentum balances of the system and the equations usually depict the space variations and time variations of the state variables. e.g., non-ideal mixing conditions in batch reactors, heat and mass transfer in packed beds etc. Hence solving PDEs with ease and efficiency is of great interest in recent times. A heat transfer chemical engineering problem i.e., two-dimensional steady state heat conduction in a slab under different boundary conditions was taken and solved using MOL. It is a semi-analytical or semi-discrete numerical method to calculate accurate numerical solutions of PDEs. The comparisons are made between analytical solutions and MOL solutions, and the result is a positive outcome [1]. MOL is superior to finite difference method (FDM) in terms of set- up time and flexibility for incorporation of conservation equations [2]. MOL generally uses a central difference approximation, followed by solving the resulting differential equations analytically based on matrix exponential method [3]. The MOL solution for the wave equation is solved by using higher order finite difference approximation gave improved accuracy and superior performance with less computational effort [4]. It is easier to apply MOL technique than few other numerical methods including finite difference and element method. The main advantage of using MOL application is its accurate results [5]. The accuracy of MOL technique can be analysed by using matrix analysis method. Modified MOL method is also applied to numerous test problems to analyse the accuracy and efficiency of both standard and modified MOL techniques [6]. Convection Diffusion problem, which is time-dependent (Singularly Perturbed problems) with time delay is solved using a numerical method, and is obtained by a step discretization problem. Finite difference scheme is applied for solving the spatial derivatives in the diffusion equation and for the time variable Crank Nicholson finite difference method is used, and the convergence of the scheme is analysed [7]. A significant class of non- linear parabolic integra-differential equations (PIDEs) are solved using an efficient numerical scheme called mixed group preserving scheme (GPS) and spectral meshless radial point interpolation (SMRPI) by the MOL is introduced [8]. Addressed one of Ostwald-de Waele fluid problems by taking into account the heat impacts which are involved in the non- Newtonian flow in a rotating disk with variable thickness. Nonlinear coupled partial differential quations which represents the non-Newtonian flow dynamics are reduced to non-linear coupled ordinary differential equations using MOL [9]. For solving different multidimensional problems such as problems on eigenvalues and eigenvectors of transition matrix for first and third type of boundary conditions are solved using applied MOL. A one-dimensional parabolic equation type is solved using MOL [10]. The analysis of plane wave scattering by PAMGS is analysed using MOL which helps discretizing the PDEs into a system of ODEs [11]. Meshless MOL method is proposed to obtain numerical solution, which is having polar coordinates. Multiquadric radial basis function (RBF) is helpful in discretizing the space variables, and by using 4^{th} order RK method. This particular work is an extension to understand the versatility of RBFs in non-rectangular coordinates, since major portion of research is already devoted to PDEs in rectangular coordinates in earlier times. This meshless numerical method has some serious advantages, which are more flexible, saves computational time and also do not require mesh generation [12].

1.1. Analytical Solution:

Analytical solutions are mathematical functions that define dependent variables as a function of independent variables. Except the simplest PDE problems, analytical solutions are generally difficult to derive mathematically and also, they are exact. As non-homogeneity in boundary conditions increases, the complexity in evaluation of analytical solution also increases for solving PDEs. But in real life problems and in chemical engineering systems mostly non-homogeneous boundary conditions exist s homogeneous boundary conditions are complex to maintain. As a consequence, alternative Numerical Difference (effective) methods are gaining importance for solving PDEs, such as MOL in the fields of science and engineering.

1.2. Method of lines (Semi-Analytical Solution):

The space derivatives are discretized by using MOL. While the time derivative is kept continuous. The eigen value method is best used for solving adjoint system of linear first order ODEs analytically. A discrete sequence of piecewise temperatures-time variations is obtained at each line as outcome of the computational procedure, and the resultant solution is described in terms of linear combinations of exponential functions containing eigen values as well as eigen vectors. The MOL embodies a hybrid procedure to solve PDEs in semi-analytical manner. Solving adjoint system ODEs is much easier than solving original PDEs using MOL technique. The required algebraic expressions of spatial derivatives in the differential equations are based on different numerical schemes for ex: finite differences, elements and collocations or Fourier series. In the differential equation, algebraic approximations are replaced in place of spatial derivatives using MOL. Once the replacement is done, only initial value variable time remains. The system of ODEs approximates to PDEs since only one remaining independent variable is present.

$$u_x = \frac{- u_{i-1}}{\Delta x}$$

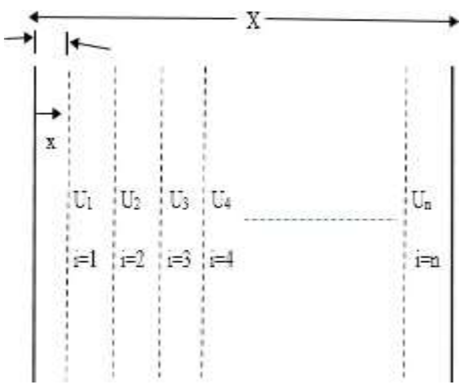

Fig.1. Schematic Diagram of MOL analysis.

The MOL is one of the most used and well-established numerical method for solving of ODEs. In order to convert the spatial derivative du_i/dt into algebraic approximation, the finite difference scheme (FD) is used.Where ‘i’ indicates a position along the grid in x-axis and Δx is the space between the two lines in x along the grid, and it is assumed as constant. Thus for the extreme left side value of x, i=1and for the extreme right side value of x, i=N and totally x has N points.

For $1 \leq i \leq N$, the MOL is: $\frac{du_i}{dt} = \frac{u_i - u_{i-1}}{\Delta x}$

For further calculation of initial conditions and boundary conditions are needed. As there is need of counting errors, truncation error is added which is calculated from truncated Taylor series.

$$\frac{du_i}{dt} = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x)$$

Where O(Δx) denotes order Δx i.e., truncation error. The numerical integration of the MODEs of equations. If the derivativeis approximated by a first order FD:

$$\frac{du_i}{dt} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

Where n is an index for the variable t (t moves forward in steps denoted or indexed by n).

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_i^n - u_{i-1}^n}{\Delta x}$$

Hence, solve for u_i^{n+1}explicitly can solve for the solution at the advanced point in t, n+1, from the solution at the base point n. This procedure is generally termed the forward Euler method which is the most basic form of ODE integration. MOL provided temperature-time solutions and the analytical solutions obtained by Eigen value method exhibits excellent quality are both are in great agreement at all times. The MOL application can be extended to solve second order hyperbolic, elliptic and mixed hyperbolic-parabolic problems. Because of these wide applications and advantages of MOL, to various engineering problems, different solvers and libraries are available in Mathematics such as MATLAB, FORTRAN, and Maple C.

2. Problem Statement:
Consider a very long square bar of 0.1*0.1m cross section. Assume that two- dimensional steady state heat conduction with no heat generation is taking place in the slab with thermal conductivity K= 1.5 W/m℃ and h= 45 W/m^2℃, to determine the temperature profile in the plate with different types of boundary conditions by using analytical and method of lines.

2. 1. Case-I: Dirichlet Boundary Conditions:
The bottom edge is maintained at 100ºC while the other three edges are maintained at 0ºC.

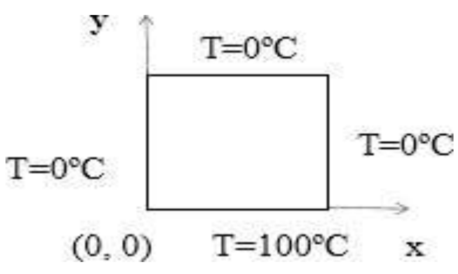

Fig. 2. 2-D Slab under Dirichlet Boundary Conditions.

Dirichlet boundary conditions:
1. x=0; T=0 ℃;
2. x=a; T=0 ℃;
3. y=b; T=0 ℃;
4. y=0; T=100^0 C.

2.2. Case-II: Neumann Boundary Conditions:
The top edge is insulated and the right edge is maintained at 100ºC while the other two edges are maintained at 0ºC.

Neumann Boundary Conditions:
1. x=0; T=100ºC;
2. x=a; T=0ºC;
3. y=0; T=0ºC;
4. y=b; $\frac{dT}{dy} = 0°C$;

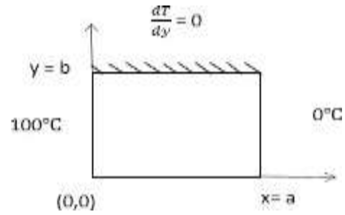

Fig. 3. 2-D Slab under Neumann Boundary Conditions.

3. Results and Discussions:

3.1. Results:

Temperature profiles for two-dimensional steady state heat conduction in a slab without internal heat generation under Dirichlet boundary conditions:

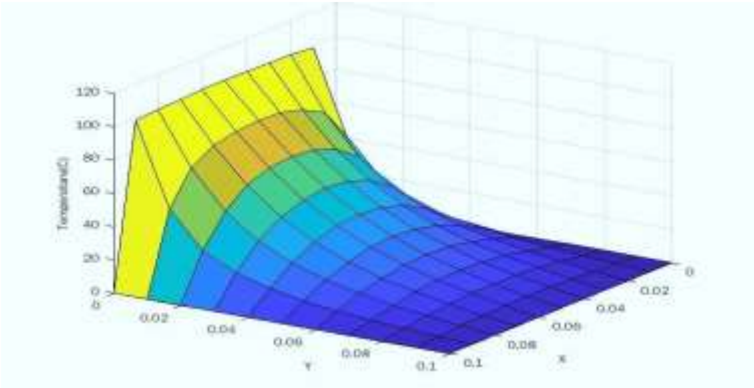

Fig. 4. Temperature distribution in a given slab by using Analytical method (Case-I) at n=10.

The temperature distribution profile is plotted for analytical method under Dirichlet boundary conditions when 'n' (number of lines/strips) is taken as 10. The specified boundary conditions for the above plot are; At y=0,T=100°C; At x=0, T=0°C; At x=0.1,T=0°C; At y=0.1,T=0°C.

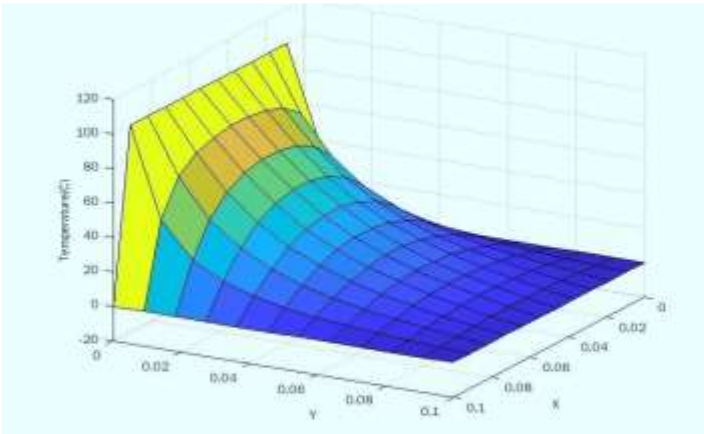

Fig. 5. Temperature distribution in a given slab by using MOL method (Case-I) at n=10.

The temperature distribution profile is plotted for MOL method under Dirichlet boundary conditions when 'n' (number of lines/strips) is taken as 10. The specified boundary conditions for the above plot are ; At y=0,T=100°C; At x=0, T=0°C; At x=0.1,T=0°C; At y=0.1,T=0°C.The x-variables are discretized into 'n' strips with respect to y-axis.

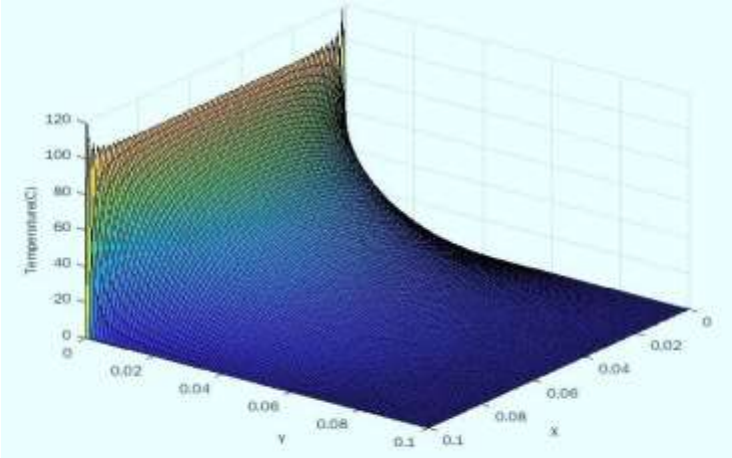

Fig. 6. Temperature distribution in a given slab by using Analytical method (Case-I) at n=100.

The temperature distribution profile is plotted for analytical method under Dirichlet boundary conditions when 'n' (number of lines/strips) is taken as 100. The specified boundary conditions for the above plot are; At y=0, T=100°C; At x=0, T=0°C; At x=0.1,T=0°C; At y=0.1,T=0°C.

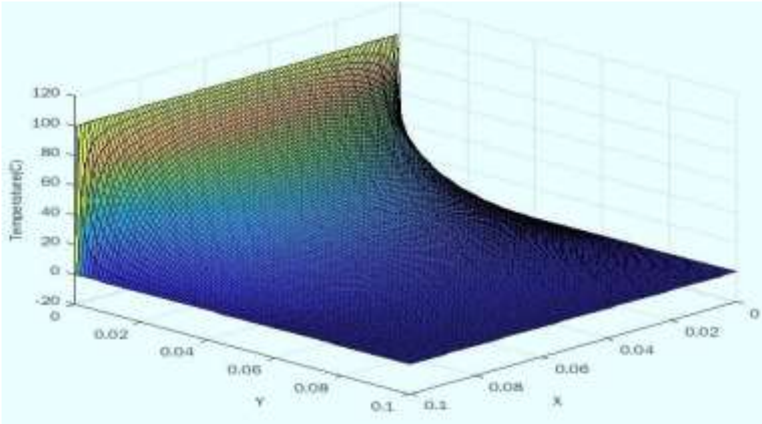

Fig. 7. Temperature distribution in a given slab by using MOL method (Case-I) at n=100.

The temperature distribution profile is plotted for MOL method under Dirichlet boundary conditions when 'n' (number of lines/strips) is taken as 100. The specified boundary conditions for the above plot are ; At y=0,T=100°C; At x=0, T=0°C; At x=0.1,T=0°C; At y=0.1,T=0°C.The x-variables are discretized into 'n' strips with respect to y-axis.

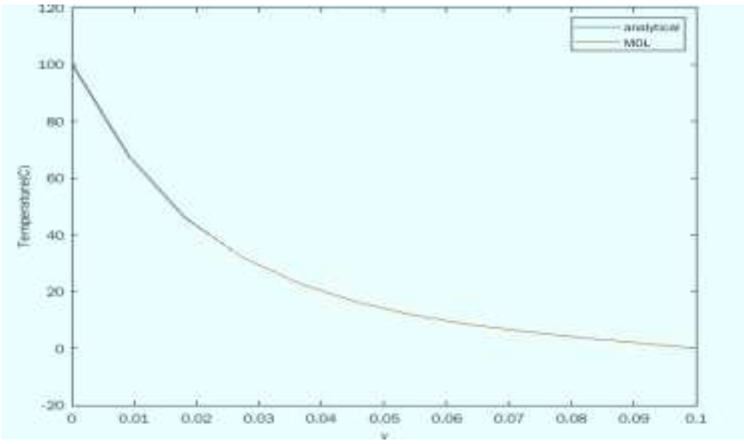

Fig. 8. Comparison between Analytical and MOL method for case-1 at x=0.02 when n=10.

For the Dirichlet boundary conditions, the above graph is plotted for comparing the solution results obtained from both analytical and MOL (Semi-Analytical) method at x=0.02 for n=10.

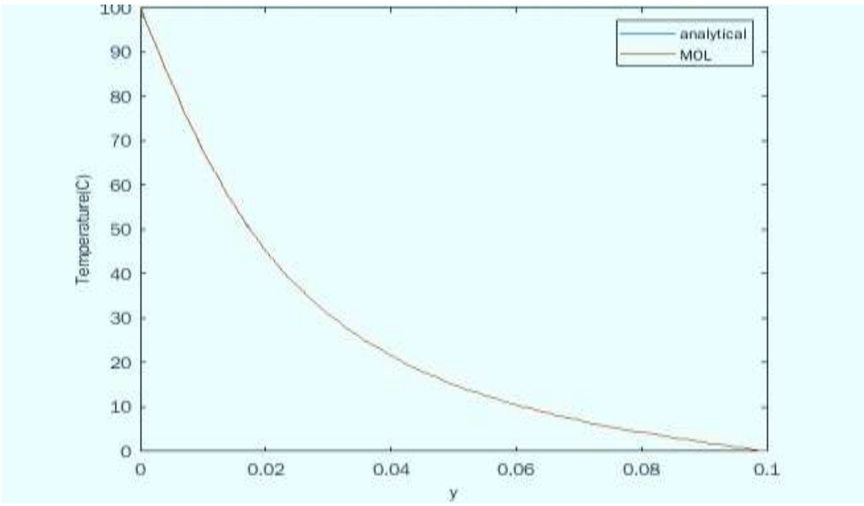

Fig. 9. Comparison between Analytical and MOL method for case-1 at x=0.02 when n=100.

For the Dirichlet boundary conditions, the above graph is plotted for comparing the solution results obtained from both Analytical and MOL(Semi-Analytical) method at x=0.02 for n=100.

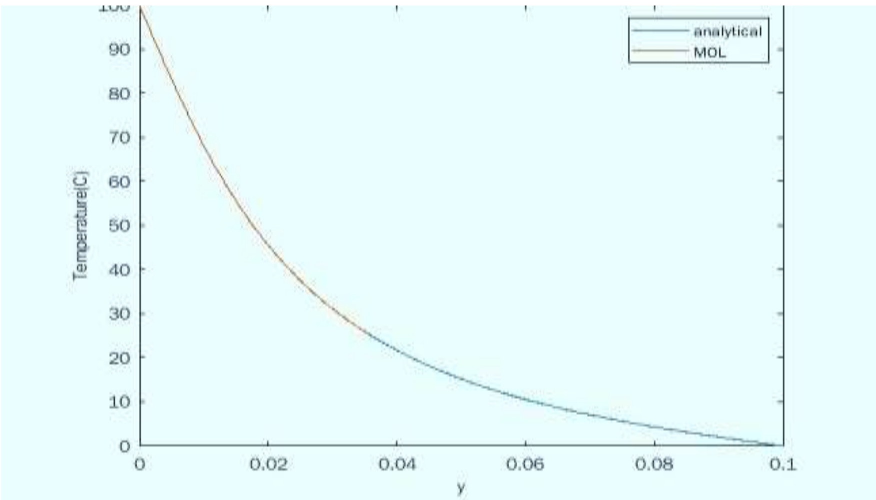

Fig. 10. Comparison between Analytical and MOL method for case-1 at x=0.02 when n=1000.

For the Dirichlet boundary conditions, the above graph is plotted for comparing the solution results obtained from both Analytical and MOL(Semi-Analytical) method at x=0.02 for n=1000.

S. No	Analytical	MOL using n=10	MOL using n=100	MOL using n=1000	Error when n=10	Error when n=100	Error when n=1000
1	99.6584	100	100	100	0.3416	0.3416	0.3416
2	68.2262	67.3594	67.9942	68.1882	0.8668	0.232	0.038
3	45.6341	45.2717	45.5426	45.6195	0.3624	0.0915	0.0146
4	31.2202	31.1236	31.1986	31.2169	0.0966	0.0216	0.0033
5	21.7757	21.7559	21.7719	21.7751	0.0198	0.0038	0.0006
6	15.2754	15.2822	15.2780	15.2759	0.0068	0.0026	0 .0005
7	10.6035	10.609	10.6048	10.6037	0.0055	0.0013	0.0002
8	7.1076	7.1122	7.1090	7.1078	0.0046	0.0014	0.0002
9	4.3659	4.3694	4.3671	4.3661	0.0035	0.0012	0.0002
10	2.0766	2.0792	2.0772	2.0767	0.0026	0.0006	0.0001
11	0	0	0	0	0	0	0

Table 1. Tabulated results of analytical and method of lines (MOL) solutions (Case-I) at x=0.02.

Temperature profiles for two-dimensional steady state heat conduction in a slab without internal heat generation under Neuman boundary conditions.

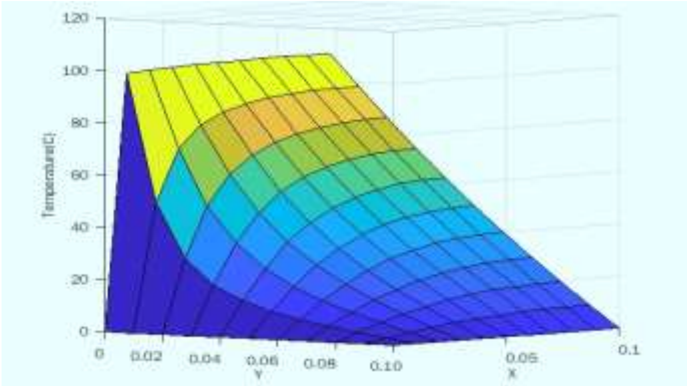

Fig. 11. Temperature distribution in a given slab by using Analytical method (Case-I) at n=10.

The temperature distribution profile is plotted for analytical method under Neumann Boundary Conditions when 'n' (number of lines/strips) is taken as 10. The specified boundary conditions for the above plot are: At y=0,T=0°C; At x=0,T=100°C; At x=0.1,T=0°C, At y=0.1, The surface is insulated(Derivative of temperature with respect to spatial gradient is zero).

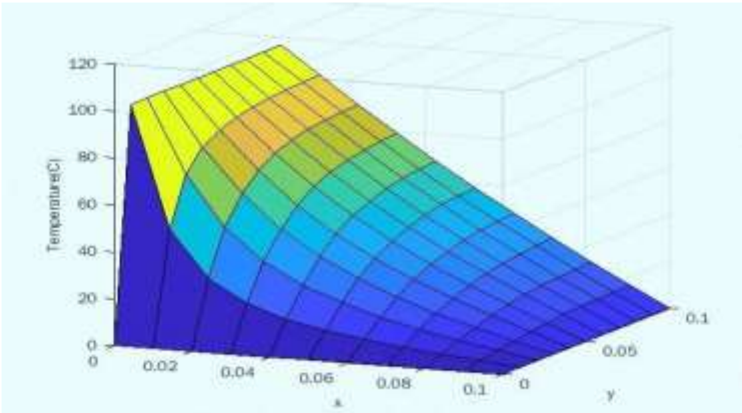

Fig. 12 Temperature distribution in a given slab by using MOL method (Case-II) at n=10.

The temperature distribution profile is plotted for MOL method under Neumann Boundary Conditions when 'n' (number of lines/strips) is taken as 10. The specified boundary conditions for the above plot are: At y=0,T=0°C; At x=0,T=100°C; At x=0.1,T=0°C, At y=0.1, The surface is insulated(Derivative of temperature with respect to spatial gradient is zero). The y- variables are discretized into 'n' strips with respect to x-axis.

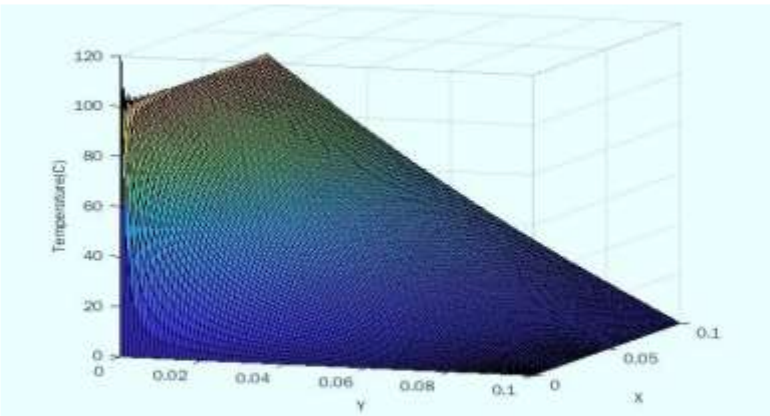

Fig. 13. Temperature distribution in a given slab by using Analytical method (Case-I) at n=100.

The temperature distribution profile is plotted for analytical method under Neumann Boundary Conditions when 'n' (number of lines/strips) is taken as 100. The specified boundary conditions for the above plot are: At y=0, T=0°C; At x=0,T=100°C; At x=0.1,T=0°C, At y=0.1, The surface is insulated(Derivative of temperature with respect to spatial gradient is zero).

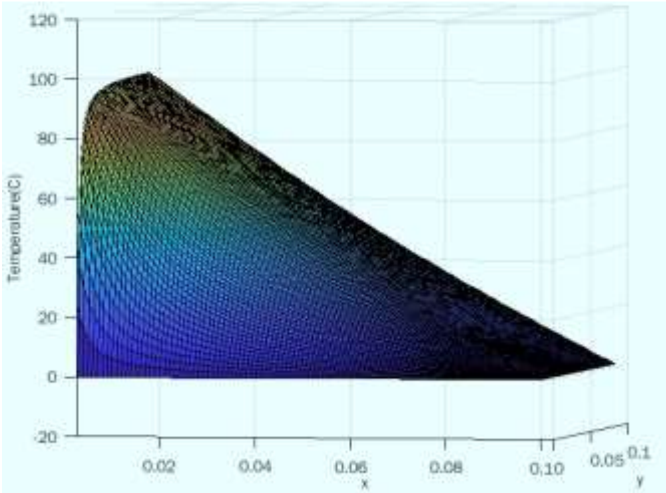

Fig. 14. Temperature distribution in a given slab by using MOL method (Case-II) at n=100.

The temperature distribution profile is plotted for MOL method under Neumann Boundary Conditions when 'n' (number of lines/strips) is taken as 10. The specified boundary conditions for the above plot are: At y=0,T=0°C; At x=0,T=100°C; At x=0.1,T=0°C, At y=0.1, The surface is insulated(Derivative of temperature with respect to spatial gradient is zero). The y- variables are discretized into 'n' strips with respect to x-axis.

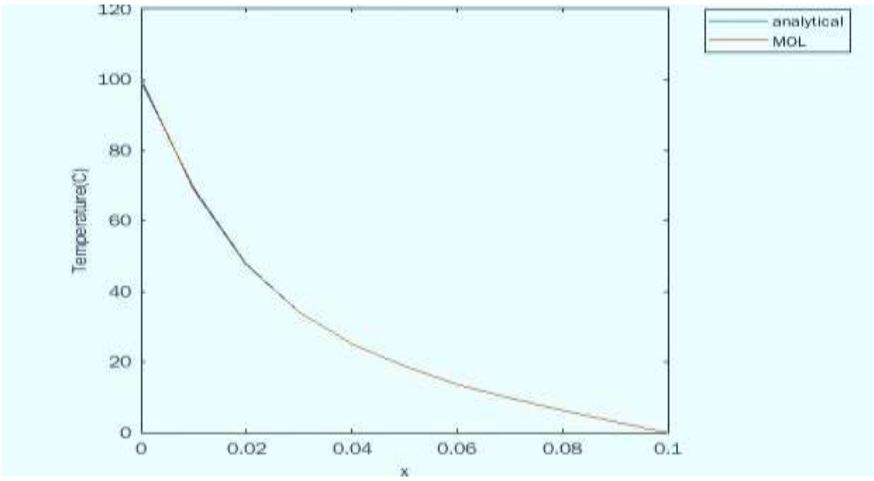

Fig. 15. Comparison between Analytical and MOL method for case-II at y=0.02 when n=10.

For the Neumann Boundary conditions, the above graph is plotted for comparing the solution results obtained from both Analytical and MOL methods at y=0.02 for 'n'=10.

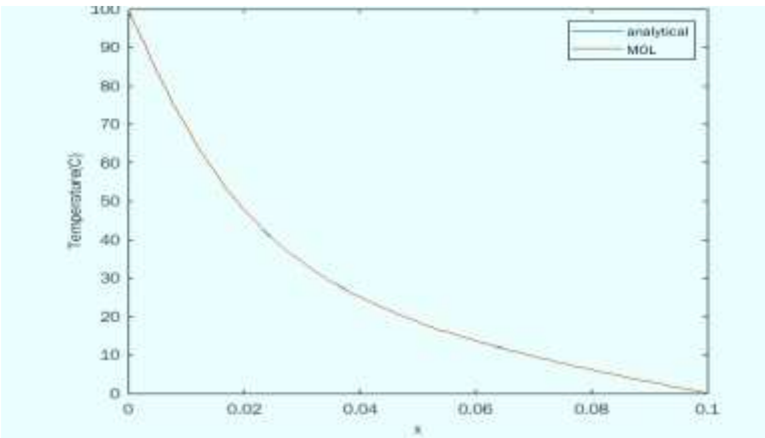

Fig. 16. Comparison between Analytical and MOL method for case-II at y=0.02 when n=100.

For the Neumann Boundary conditions, the above graph is plotted for comparing the solution results obtained from both Analytical and MOL methods at y=0.02 for 'n'=100.

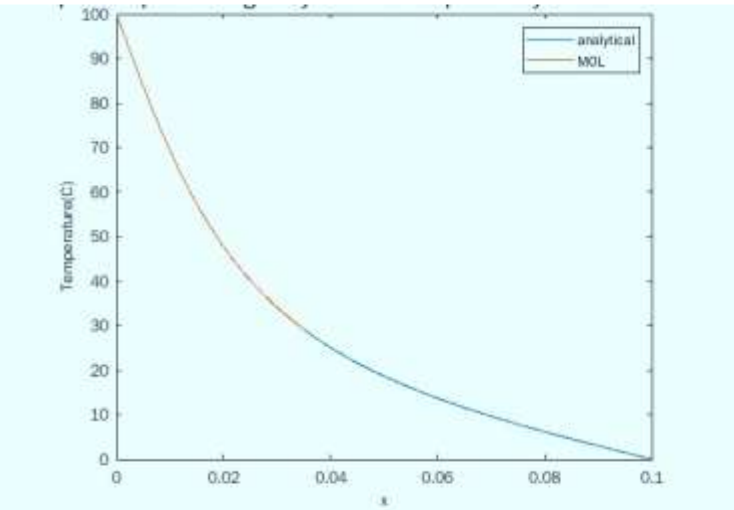

Fig. 17. Comparison between Analytical and MOL method for case-II at y=0.02 when n=1000.

For the Neumann Boundary conditions, the above graph is plotted for comparing the solution results obtained from both Analytical and MOL methods at y=0.02 for 'n'=1000.

S.No	Analytical	MOL using n=10	MOL Using n=100	MOL using n=1000	Error when n=10	Error when n=100	Error When n=1000
1	99.6584	100	100	100	0.3416	0.3416	0.3416
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6	15.2754	15.2822	15.2780	15.2759	0.0068	0.0026	0 .0005
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9	4.3659	4.3694	4.3671	4.3661	0.0035	0.0012	0.0002
10	2.0766	2.0792	2.0772	2.0767	0.0026	0.0006	0.0001
11	0	0	0	0	0	0	0

Table 2. Tabulated results of analytical and method of lines (MOL) solutions (Case-II) at y=0.02.

3.2. Discussions:

The temperature distribution profiles in the two-dimensional steady state heat conduction in a slab are plotted with the profiles of analytical solutions by using MATLAB. The steady state analysis of temperature distribution with specified Dirichlet and Neumann Boundary conditions was developed by using Method of lines.

1. From the graphs it is observed that the temperature profiles obtained from the method of lines are similar to that of the analytical method, showing that it can achieve an approximately accurate temperature profile using the Method of lines.
2. From comparison of graphs and tables for both boundary conditions, it is clear that as the number of lines increasing, the error between analytical and method of lines (semi analytical) solutions has been decreased, and the profiles of analytical and MOL Converged which indicates the preference of higher number of lines in order to obtain accurate values (with minimal errors).

4. Conclusion:

In this study and review on, a two-dimensional slab with steady state heat conduction under different boundary conditions was modeled. The temperature distribution profiles for both analytical and semi analytical methods under Dirichlet and Neumann boundary conditions were plotted. For the semi analytical approach, it was solved by using Method of Lines. The effect of the number of lines on the convergence of solution from method of lines (semi analytical method) to analytical solution was studied under graphical and individual nodal data analysis in a detailed way. It is observed that the two methods converge and give almost similar data with minimal error. As the non-homogeneity in the boundary conditions increases, solving analytically becomes more complex, also time taking, for such non-homogeneous boundary conditions it is assumed that the MOL doesn't much increase the complexity.

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