"Geometry Involving Generalized Fixed Point Theory In Banach Spaces"

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Abstract :
This paper explains the convergence results of common best proximity points of various of non-self mappings in the setting of Banach spaces. In detail, we approximate new line a common fixed point for the class of relatively non expansive mappings by using fixed point . Through this result, we approximate common best the help of projective operators. And, we prove the new line convergence results of geometry involving in Banach spaces, generalized non expansive new line and proximally quasi-contractive multivalued mappings in uniformly convex Banach.

Keywords : Fixed point; Quasimodo; Uniform convexity; Uniform convexity in every direction; Asymptotic centre; Property UC; k-uniform con verity; geometry ; mappings .

Introduction: In this paper we list some geometric properties of Banach spaces and some fixed point theory results which we use in the subsequent title. Certain geometric properties of a Banach space X are helpful to study the existence of fixed points for non expansive mappings and asymptotically non expansive mappings on closed bounded convex subsets of X. In this paper we give a brief introduction to Geometry of Banach spaces, a brief introduction to fixed point theory, a brief introduction to best proximity point theorems, and finally we give contents of the paper.
Let X be a normed linear space with norm ||·||. We denote the closed unit ball of X by B_x, which is defined by B_x := {x ∈ X : $\|x\|$ ≤ 1} and the closed unit sphere of X by S_x which is defined by
S_x := {x ∈ X : $\|x\|$ = 1}. For a nonempty subset M of X, x ∈ X and δ > 0, let
PM(x) := {y ∈ M : $K_x - K_Y$ = dist(x, M)}; P_M (x, δ) :
= {y ∈ M : $K_x - K_y$ ≤ dist(x, M) + δ}.

The set P_M (x) is called the set of all best approximations of M to x. The set P_M (x, δ) is always nonempty whereas the set P_M (x) need not be nonempty. The set M is said to be proximal at x if P_M (x) is nonempty and Chebyshev at x if P_M (x) is singleton. A sequence (x_n) is said to be a minimizing sequence of M at x if $kx_n - x_k$ → dist(x, M).
The set M is said to be approximatively compact at x if every minimizing sequence of M at x has a convergent subsequence in M. The set M is said to be proximal, Chebyshev and approximatively compact if it is proximal, Chebyshev and approximatively compact at every x ∈ X respectively. We will see that there are nice relations between some geometric properties of X and approximation properties of closed convex subsets of X. A normed linear space X is said to be strictly convex if it does not contain a nontrivial line segment on its unit sphere and X is said to have KK property if the weak convergence and strong convergence are equivalent on its unit sphere.
We obtain some results for order-preserving continuous maps without monotone non expansive condition in a uniformly convex Banach space having monotone norm. Under similar situation we also obtain a fixed point theorem in a reflexive Banach lattice. Throughout this chapter, a Banach space is over K which is either the filed R of real numbers or the filed C of complex numbers unless otherwise specified. The norm on a Banach space is denoted by ||. ||

Asymptotically non expansive and monotone asymptotically non expansive mapping :
Definition 1. Let X be a Banach space and K be a nonempty subset of X. A map T : K → K is said to be
1. Non expansive if ||T(x) − T(y)k ≤ K_x − y || for every x, y ∈ K,
2. asymptotically non expansive if there exists a sequence (k_n) of positive real numbers with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n(x) - T^n(y)\| \le k_n \|x - y\|$
for every x, y in K and n = 1, 2, ... 3. pointwise asymptotically non expansive if for each x ∈ K there exits a sequence α_n (x) of positive real numbers which converges to 1, such that $\|T^n(x) - T^n(y)\| \le \alpha_n$(x) $k_x - y_k$ for each integer n ≥ 1 and for each y in K
Hence non expansive mappings are also asymptotically non expansive mappings and asymptotically non expansive mappings are also pointwise asymptotically non expansive mappings.

Definition 2. A Banach space X is said to be uniformly convex (UC), if for every ∈ with 0 < ∈ ≤ 2

$$\boldsymbol{\delta}(\epsilon) := \{\mathbf{1} - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\} > 0$$

The function δ is called modulus of convexity of the Banach space X. It is known that δ is an increasing continuous function on (0, 2]. This fact is used to prove .

Definition 3. A nonempty subset M of X is said to be k-strongly Chebyshev at x ∈ X if M is approximatively compact at x and $diam_K(P_M$(x, 0)) = 0.

Definition 4. For k ∈ Z^+ , the space X is said to be k-strongly convex if X is reflexive, k-rotund and it has KK property. This notion of k-strongly Chebyshev coincides with strongly Chebyshev when k = 1. If M is k-strongly Chebyshev at every element of a set N then we say that M is k-strongly Chebyshev on N. If M is k-strongly Chebyshev on X then we say M is k-strongly Chebyshev.

An Introduction to Geometry of Banach Spaces :

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P_M(x) := {y ∈ M : $\|x-y\|$ = dist(x, M)};

P_M (x, δ) := {y ∈ M : $\|x-y\|$ ≤ dist.(x, M) + δ}.

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The set P_M (x, δ) is always nonempty whereas the set P_M (x) need not be nonempty. The set M is said to be proximal at x if P_M (x) is nonempty and Chebyshev at x if P_M (x) is singleton. A sequence (x_n) is said to be a minimizing sequence of M at x if $\|x_n - x\|$ → dist(x, M). The set M is said to be approx. amatively compact at x if every minimizing sequence of M at x has a convergent subsequence in M. The set M is said to be proximal, Chebyshev and approximatively compact if it is proximal, Chebyshev and approximatively compact at every x ∈ X respectively. We will see that there are nice relations between some geometric properties of X and approximation properties of closed convex subsets of X. A normed linear space X is said to be strictly convex if it does not contain a nontrivial line segment on its unit sphere and X is said to have KK property if the weak convergence and strong convergence are equivalent on its unit sphere.

Definition . A real normed linear space X is said to be k-rotund (k-R) if $x_1, x_2, \ldots\ldots x_{k+1}$ ∈ S_x with $V(x_1, x_2, \ldots\ldots x_{k+1} > 0)$, then

$$\left\|\frac{x_1, x_2, \ldots\ldots x_{k+1}}{k+1}\right\| < 1.$$

Sullivan [13] introduced the concept of k-uniform rotundity (k ∈ Z^+) in a real normed linear space as a generalization of uniform convexity and k-strict convexity.

Theorem 1. A normed linear space X is reflexive if and only if every closed convex subset of X is proximal.

If X is reflexive then every bounded sequence has a weakly convergent subsequence. Using this fact, it is easy to see that in a reflexive space, every closed convex set is proximal. For the converse we use the following characterization of reflexivity due to James as stated below.

Theorem 2. (James). A Banach space X is reflexive if and only if every bounded linear functional on X is norm attaining.

Note that a linear functional f on X is said to be norm attaining if there exists x ∈ B_x such that f(x) = $\|.\|$ One can verify that a normed linear space X is strictly convex if and only if the set P_M(x) contains atmost one point for every closed convex subset M of X and x in X.

Theorem 3. A normed linear space X is strictly convex and reflexive if and only if every closed convex subset of X is Chebyshev.

Theorem 4. A normed linear space X is reflexive and KK if and only if every closed convex subset of X is approximatively compact. A normed linear space X is said be strongly convex if X is reflexive, strictly convex and it has KK property.

Clarkson introduced a notion called uniform convexity in normed linear spaces. Uniformly convex spaces unify inner produced spaces and L_p spaces for 1 < p < ∞.

Theorem 4. Let X be a real Banach space. The following statements are equivalent.

1. X is uniformly convex.
2. B_x is uniformly strongly Chebyshev on X.
3. B_x is uniformly strongly Chebyshev on $2S_X$.
4. S_X is uniformly strongly Chebyshev on rS_X for every r > 0.
5. S_X is uniformly strongly Chebyshev on $\frac{1}{2}S_X$.
6. $(S_X, \frac{1}{2}S_X)$ has property UC.

In [19] the authors have given some characterizations of strong convexity in terms of property UC as follows.

Theorem 5. (Nadler). Let (X, d) be a complete metric space and T : X → C_B(X) be a multivalued contraction. Then there exists x_0 ∈ X such that x_0 ∈ T(x_0).

Nadler's theorem gives the existence of a fixed point for a multivalued contraction map. Note that the multivalued contraction map need not have unique fixed point. Let D be a nonempty closed convex subset of a Banach space X.
If we assume S : D → D is a continuous map such that S(D) is contained in a compact set, then Theorem 4 ensures that S has a fixed point. If we assume S : D → X is a continuous map such that S(D) is contained in a compact set, then S need not have a fixed point or if we assume T : D → X is a contraction map then T need not have a fixed point. In 1955, Kwasniewski [14] proved a fixed point theorem combining two fixed point theorems namely, Banach fixed point theorem and Schauer fixed point theorem which gives the existence of a fixed point for the sum of two maps.
Theorem 6. Let D be a nonempty closed convex subset of a Banach space X and S, T : D → X be maps such that
(i)T is a contraction map;
(ii) S is continuous and S(D) is contained in a compact set;
(iii) T(D) + S(D) ⊆ D.
Then there exists x_0 ∈ D such that T x_0 + Sx_0 = x_0.
When S = 0, Theorem 1 reduces to Banach contraction theorem and when T = 0 Theorem1. reduces to Schauer fixed point theorem. In 2004, Ran and Reu rings [4] proved an analogue version of Banach contraction theorem in a partially ordered complete metric space as stated below. Later many authors [17,18,19,20] studied this result in many directions.
Lemma : Let K be a nonempty subset of a normed linear space X. A map T : K → K is said to be non expansive if $\|T_x - T_y\| \le \|x - y\|$, for all x, y ∈ K.
It is easy to see that a non expansive map on a compact set need not have a fixed point. To prove the existence of fixed points for non expansive mappings on closed bounded convex sets, we mainly focus on some geometric properties of the normed linear space X. Fixed point theory of non expansive maps and asymptotically non expansive maps is to determine the conditions on the structure of the set K or on the space X which guarantee the existence of fixed points for non expansive mappings and asymptotically non expansive mappings. In 1965 Browder [10] and Godha [16] proved independently the existence of fixed points for non expansive mappings in uniformly convex Banach spaces as stated below.
Fixed Point Theory :
Let D be a nonempty subset of X. Let T be a map from D to Y , where Y is a subset of X. In fixed point theory we mainly search for some suitable conditions on X, D or T such that the equation T_x = x has a solution for some x in D. Such a point x is said to be a fixed point of T. Now we discuss some applications of fixed point theory.
1. **Picard's Existence** and **uniqueness Theorem (in ordinary differential equations):** Let us consider the following first order differential equation.
$\frac{dx}{dt}$ = f(t, x(t))
with the initial condition x(t_0) = x_0 .
Suppose f is a real valued continuous function on
D := {(t, x) : |t − t_0| ≤ a, |x − x_0| ≤ b}
for some a, b > 0 and hence there exists c such that |f(t, x)| ≤ c for all (t, x) ∈ D. Suppose there exists a constant k such that
|f(t, x) − f(t, y)| ≤ k|x − y| for (t, x),(t, y) ∈ D.
Using Banach contraction theorem, the initial value problem has a unique solution on an interval [t_0 − α, t_0 + α], where α < min $\left\{a, \frac{b}{c}, \frac{1}{k}\right\}$.
2. **Invariant subspace problem:** The invariant subspace problem in operator theory is one of the most famous unsolved problems. If X is a Banach space and T is a bounded linear operator on X, then T has a nontrivial closed invariant subspace or not is not known for a long time. In this direction in 1987 **Enfold gave** an example of an operator in a Banach space which has no nontrivial closed invariant subspace. Though the problem is still open in Hilbert spaces, in 1973 Lomonosov proved the following theorem.

PRELIMINARY CONCEPTS AND SOME FIXED POINT RESULTS :

During last three decades the fixed point theory is developed in different ways by various mathematicians by either enriching the space structure and relaxing the mapping condition or vice -versa. Hereby we restrict our study in geometry , Banach, 2- fixed point and 3- Banach spaces and their applications. In the context of the results we obtain in subsequent chapters, we need some fundamental concepts and the fixed point results already obtained by the authors mentioned in introduction & theorems , lemma In this section we give preliminary definitions and examples related to fixed point theory.

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