Some Fixed Point Theorem And Fixed Point Iteration.

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Abstract
In this paper some fixed point theorems is established. A numbers of authors have define contraction mapping which are generalization of the well known Banach contraction. Some Fixed Point Iteration are established with examples.

Keywords. Metric spaces, common fixed point, contraction mapping.

Introduction
Fixed point theory aims to understand the existence and uniqueness and of fixed points and their applications in various areas of mathematics and other disciplines. There are many fixed point theorems whose applications provided powerful mathematical tools.In 1922, Banach created a fundamental result called Banach contraction principle in the concept of the fixed point theory. Later most of the authors intensively introduced many works regarding the fixedpoint theory in various of spaces.
The Banach fixed-point theorem also known as the contraction mapping theorem or contractive mapping theorem or Banach–Caccioppoli theorem) is an important in the theory of metric spaces. It guarantees the existence and uniqueness of fixed point of certain self-maps of metric spaces and provides a constructive method to find those fixed points. Huang and Zhang (2007) generalized the concept of metric spaces by replacing the set of real number with ordered Banach spaces and obtained some fixed point theorems of contraction mappings in cone metric spaces.

Definition. A map $T: X \to X$is said to be a contraction mapping if there exists a real number

$c < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq cd(x, y)$$

Theorem : Banach Fixed Point Theorem.
If X is complete and $T: X \to X$ is a contraction mapping then T has a unique fixed point in X. Proof. Uniqueness is easy: If x and y are both fixed points of T,
Then

$$d(x, y) = d(Tx, Ty) \leq 1cd(x, y)$$

which can happen only
if $d(x, y) = 0$, because $c< 1$, whence $x = y$.
To prove existence start with a point $x_0 \in X$.
Put $x_1 = Tx_0$. Put $x_2 = Tx_1$.
In general for any positive integer n, put $x_n= Tx_{n-1}$.
This defines a sequence $\{x_n\}$ in X. We claim that this sequence is Cauchy.
Indeed, if $m<n$, then

$$d(x_m, x_n) = d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + - - - d(x_{n-1}, x_n)$$

$$= (1 + c + - - - c^{n-m-1})d(x_m, x_{m+1}) \leq \frac{1}{1-c} d(x_m, x_{m+1})$$

$$= \frac{1}{1-c} c^m d(x_0, x_1) \to 0 \; as \; m \to \infty$$

Since X is complete, there exists $x \in X$ such that $x_n \to x$. Since T is a contraction mapping, from the triangle inequality, we have

$$d\big(x, T(x)\big) \leq d(x, xn) + d(xn, T(x))$$
$$= d(x, xn) + d(T(x_{n-1}), T(x))$$
$$\leq d(x, xn) + \alpha d(x_{n-1}, x) \to 0 \ as \ n \to \infty$$

Hence $d(x, T(x)) = 0$
This gives $T(x) = x$

Rational function

Definition: A Rational function is the ratio of two polynomial functions $R(x) = \frac{P(x)}{Q(x)}$
Where $P(x)$ and $Q(x)$are polynomial functions and $Q(x) \neq 0$.
A point z_0 is called a fixed point if $R(z_0) = z_0$.
$R(z)$,$R^2(z) = R(R(z))$,$R^3(z) = R^2(R(z))$, $- - -R^n(z) = R(R^{n-1}(z))$
A fixed point z is called

i) An attracting fixed point if $|R'(z)| < 1$
ii) A repelling fixed point if $|R'(z)| > 1$
iii) A indifferent fixed point $|R'(z)| = 1$

$$R'(z) = \lim_{y \to z} \frac{R(z) - R(y)}{z - y}$$

$$|R'(z)| \sim \left|\frac{R(z) - R(y)}{z - y}\right|$$

$$|R'(z)| \, |\boldsymbol{z} - \boldsymbol{y}| \sim |\boldsymbol{z} - \boldsymbol{R}(\boldsymbol{y})|$$

Since $|z - R(y)| < \alpha|z - y|$ where$\alpha < 1$
$R(y)$ gets attracted to z.

Theorem: A rational function of degree d has precisely $d + 1$ fixed points.
Example :

$R(z) = z^2$
$R'(z) = 2z$

Here the fixed points are 0,1,∞

$|R'(z)| =$ **0<1 attracting**
$|R'(1)| =$ **3>1 repelling**

Fixed Point Iteration

Introduce a method to find a fixed point of a continuous function g.
Initially start with p_0. Iteratively define a sequence p_n by $p_{n+1} = g(p_n)$

If $p_n = p$ then

$$p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} g(p_{n-1}) = g\left(\lim_{n \to \infty} p_{n-1}\right) = g(p).$$

Example:

$$\text{Since } x^3 + 3x^2 - 7 = 0$$
$$\Leftrightarrow 3x^2 = 7 - x^3$$
$$\Leftrightarrow x^2 = \frac{7 - x^3}{3}$$
$$\Leftrightarrow x = \pm\sqrt{\frac{7 - x^3}{3}}$$

$$\Leftrightarrow \boldsymbol{x^2} = \frac{\boldsymbol{7 - 3x^2}}{\boldsymbol{x}}$$

$$\Leftrightarrow - - - -$$

Algorithm of finding the fixed point iteration for finding the root.
Considering the several g

$$g_1(x) = x - (x^3 + 3x^2 - 7)$$

$$g_2(x) = \sqrt{\frac{7}{x} - 3x}$$

$$g_3(x) = \sqrt{\frac{7 - x^3}{3}}$$

$$g_4(x) = \sqrt{\frac{7}{3 + x}}$$

$$g_5(x) = x - \frac{x^3 + 3x^2 - 7}{3x^2 + 6x}$$

All these g have same Fixed point but g_3, g_4 and g_5 converges but g_1 and g_2 does not converges.

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