Eccentric Neighbourhood Topological Indices Of Helm Related Graphs

S. Durai Raj[1*] and K.P.L. Ajai Krishna[2]

[1 *]Principal and Associate Professor (Rtd), Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil – 629003, Tamil Nadu, India.
[2] Research Scholar, Registration No.: 21213132091001, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil – 629003, Tamil Nadu, India.
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli – 627012, Tamil Nadu, India.
E-mail: [1] durairajsprincpkc@gmail.com, [2] ajaikrishna1473@gmail.com

Abstract
Let G be a simple graph. The eccentricity $e(v)$ of a vertex v is the maximum distance between v and any other vertex u in G. We denote $S_{\{en\}(u)} = \sum_{v\in N(u)} \mathcal{E}(v)$ be the eccentricity neighbourhood $v \in N(u)$ degree of u. In this article, we found the formula to calculate some eccentric neighbourhood Topological indices of line graph, middle graph and total graph of Helm graph.

Keywords: Eccentricity neighbourhood degree, Eccentric neighbourhood topological indices, line graph, middle graph, total graph, Helm graph.

MSC: 05C12, 05C90, 05C76.

1. Introduction

Graph theory is applied in the field of molecular structure, represent as an interdisciplinary science which called as chemical graph theory or molecular topology. By the means of graph theory, we identified the structural features involve structure property activity relationships. Molecules and modelings unknown structures has been classified by characterization of chemical structure topologically have the desire properties. In the last 10 years, lot of research works have been conducted. Numerical value connected with chemical constitutions of chemical bond structures have many physicochemical properties or chemical reactivity or biological activity. The topological indices been introduced by the basis of transforming molecular graph to a numeric value which characterize the topology of the molecular graph. Molecules and molecular compound are been modeled by molecular graphs. By a molecular graph, we mean a simple graph with vertices are often used for atoms and edges where used as bonds. This can be note as many ways by means of drawing, by means sequence of numbers, by means of matrix, by means of polynomial or by means of a derived number which named as topological index. So a numberic value associated by a graph is called as topological index. Some popular topological indices are distance-based, counting related and degree based topological indices. The eccentric neighbourhood degree based first, second and third Zagreb index are of the most important. In this research, a graph we mean an undirected, simple and finite connected graph. Let $G = (V(E), E(G))$ be a graph, where $V(G)$ and $E(G)$ denoted by the vertex set and edge set respectively. The set $N_G(v)$ or $N(v)$ of all neighbors of v is called the open neighbourhood of v. That is, $N(v) = \{u \in V(G) | uv \in E(G)\}$. The degree $d_G(v) = d(v)$ of a vertex v in G is defined as $d(v) = |N(v)|$. The length of the shortest path connecting between two vertices u and v is termed as the distance between two vertices and is denoted by $d_{\{G\}}(u, v)$ or $d(u, v)$.

The vertices belonging to a edge are called the ends or end vertices of the edge. Let W_{n+1} be the wheel graph, a wheel graph is a graph formed by connecting a single vertex (v) to all vertices of a cycle $(v_i, 1 \le i \le n)$. The line graph, [22] of the graph G, written as $L(G)$, is the simple graph whose vertices are the edges of G, with $vu \in E(L(G))$ when v and u have a common end point in G. An extension of the idea of line graph is the graph valued function referred as a total graph. The total graph [22] of G, denoted by $T(G)$, is the graph whose adjacent vertices corresponds to the union of the set of vertices and edges of G, with two vertices of $T(G)$ is being adjacent if and only if the corresponding element are adjacent or incident in G. The middle graph [22] of $G(V, E)$, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is a vertex and the other is an edge incident with it and it is denoted by $M(G)$. A numeral can uniquely identify a graph.

The origin of the topological indices goes back to 1947 by a chemist Wiener named the first topological index, recognize as Wiener index [1], used for searching the boiling points and which is defined as $W(G) = \frac{1}{2}\sum_{uv \in E(G)} d(u, v)$.

Among the topological indices defined in the starting phase, the Zagreb indices are related to most common molecular descriptors. Gutman and Trinajstic [2] was first introduced the first, second and third Zagreb indices given as follows:

$$M_1(G) = \sum_{v\in V(G)} d_G^2(v) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$$
$$M_3(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$$

For more details of those indices see [3][4][5].
The eccentricity defined by $\mathcal{E}(v) = \max_{u\in V(G)} d_G(u,v)$. Also the radius $r(G)$ and diameter $D(G)$ of G are defined by $r(G) = \min_{v\in V(G)} \mathcal{E}(v)$ and $D(G) = \max_{v\in V(G)} \mathcal{E}(v)$ respectively.
The eccentric connectivity index $E(G)$ in [6] defined by $E(G) = \sum_{u\in V(G)} d_G(u)\mathcal{E}_G(u)$. The eccentric connectivity index applications we see [7][8][9] and for the mathematical properties of eccentric connectivity index see [10][11][12]. we denote $S_{en}(u) = \sum_{v\in N(u)} \mathcal{E}(v)$ $ be the eccentricity neighborhood degree. H.Ahmed et al in [13] introduced the first, second and third eccentric neighborhood Zagreb indices are given as follows:

$$E_N M_1(G) = \sum_{u\in V(G)} S_{en}^2(u) \qquad \text{……………………………………} (1.1)$$

$$E_N M_2(G) = \sum_{uv\in E(G)} S_{en}(u)\, S_{en}(v) \qquad \text{……………………………………} (1.2)$$

$$E_N M_3(G) = \sum_{uv\in E(G)} (S_{en}(u) + S_{en}(v)) \qquad \text{……………………………………} (1.3)$$

Continuing with this research trajectory, S. Wazzan and H. Ahmed in 2023 [13], the chemical importance and applications of eccentric Zagreb indices are found in [23][6] In [24], Aslam et. Al gave formula to find some degree based topological invariants of boron triangular nanotube, which is ordered stack of wheel graph. In the following seconds, we find the formula to calculate the eccentric neighbourhood Zagreb indices of line graph, middle graph and total graph of Helm graphs.

2. Main Results

2.1. Eccentric Neighbourhood Zagreb indices of Helm graph $H_n, n \geq 4$

Let G be a Helm graph H_n obtained from a wheel graph W_{n+1} by adjoining a pendant edge at each vertex of the cycle. The graph H_6 is shown in Figure 1

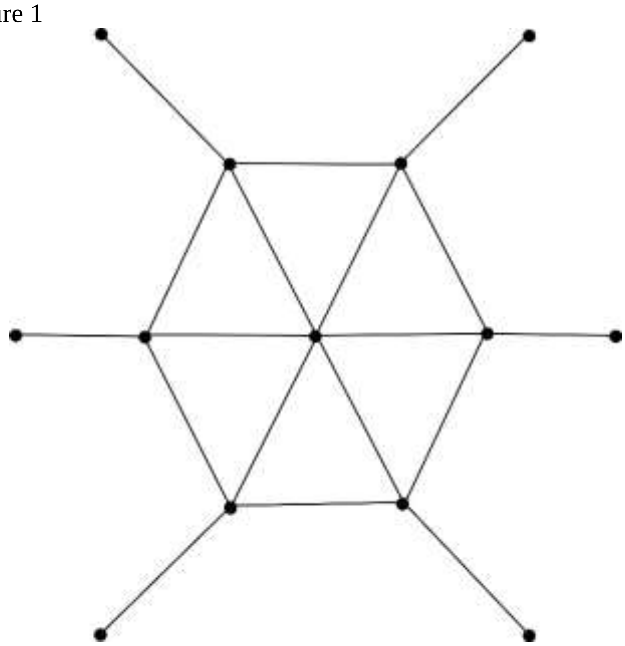

Figure 1: The Helm graph H_n

$S_{en}(u)$, where $u \in V(H_n)$	3	12	$3n$
Number of vertices	n	n	1

Table 1: Vertex partition of H_n

$(S_{en}(u), S_{en}(v))$, where $uv \in E(H_n)$	$(3,12)$	$(12,12)$	$(12,3n)$
Number of edges	n	n	n

Table 2: Edge partition of H_6

Theorem 2.1.
Let $G = H_n$ $(n \geq 6)$ be a Helm graph. Then
1. $E_N M_1(G) = 3n(3n + 51)$
2. $E_N M_2(G) = 36n(n + 5)$
3. $E_N M_3(G) = 3n(n + 17)$

Proof
The Helm graph $G = H_n$ consists of $2n + 1$ vertices and $3n$ edges. Based on the eccentric neighbourhood degree sum of vertices of G we partition $V(G)$ into subsets as shown in Table 1 and also we partition $E(G)$ based as the eccentric neighbourhood degree sum of end vertices of edges in G as shown in Table 2. Using formulae (1.1) – (1.3) to this information from Table 1 and Table 2, we obtained the required result.

2.2. Eccentric neighbourhood Zagreb Indices of Line graph of Helm Graph (H_n), $n \geq 3$

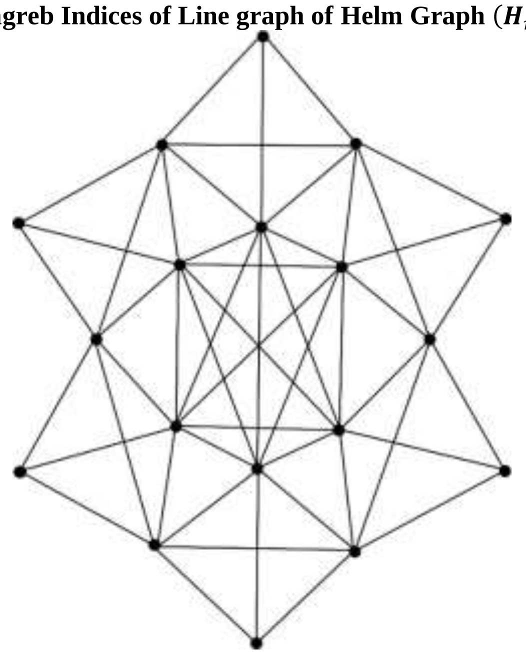

Figure 2: The line graph $L(H_6)$

$S_{en}(u)$, where $u \in V(L(H_n))$	8	16	$2n + 7$
Number of vertices	n	n	n

Table 3: Vertex partition of $L(H_n)$, $n \geq 6$

$(S_{en}(u), S_{en}(v))$, where $uv \in E(L(H_n))$	$(8,16)$	$(8,2n + 7)$	$(16,16)$	$(16,2n + 7$	$(2n + 7,2n + 7)$
Number of edges	$2n$	n	n	$2n$	$\frac{n(n-1)}{2}$

Table 4: The edge partition of $L(H_n)$, $n \geq 6$

Theorem 2.2. Let G be the line graph of Helm graph (H_n), $n \geq 6$. Then
1. $E_N M_1(G) = n(4n^2 + 28n + 369)$
2. $E_N M_2(G) = n(2n^3 + 12n^2 + \frac{181n}{2} + \frac{1535}{2})$
3. $E_N M_3(G) = n(2n^2 + 11n + 140)$

Proof
The line graph G of Helm graph H_n has $3n$ vertices and $\frac{n(n+11)}{2}$ edges. Based on the eccentric neighbourhood degree sum of vertices of G we partition $V(G)$ into subsets as shown in Table 3 and also we partition $E(G)$ based on the eccentric

neighbourhood degree sum of end vertices of edges in G as shown in Table 4. Using formulae (1.1) – (1.3) to this information from Table 3 and Table 4, we obtained the required result.

2.3. Eccentric Neighbourhood Zagreb indices of middle graph of Helm graph $H_n, n \geq 4$

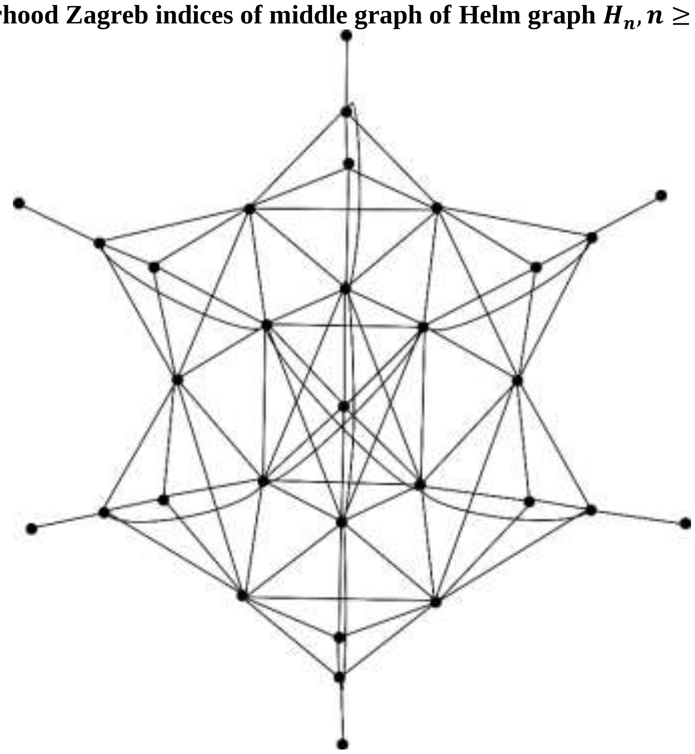

Figure 3: Middle graph $M(H_6)$

$S_{en}(u)$, where $u \in V(M(H_n))$	$3n + 16$	4	20	15	30	$3n$
Number of vertices	n	n	n	n	n	1

Table 5: The vertex partition of $M(H_n)$

$(S_{en}(u), S_{en}(v))$, $uv \in E(M(H_n))$	(4,20)	(15,20)	(20,30)	(15,30)	(30,30)	(15, $3n + 16$)	(20, $3n + 16$)	(30, $3n + 16$)	($3n + 16$,$3n + 16$)	($3n$,$3n + 16$)
No.of edges	n	n	$2n$	$2n$	n	n	n	2n	$\frac{n(n-1)}{2}$	n

Table 6: The edge partition of $M(H_n)$

Theorem 2.3. Let G be the middle graph of Helm graph $H_n, n \geq 4$. Then
1. $E_N M_1(G) = n(9n^2 + 105n + 1797)$
2. $E_N M_2(G) = n(\frac{9}{2}n^3 + \frac{9}{2}n^2 - 209n + 4724)$
3. $E_N M_3(G) = n(3n^2 + 202n + 488)$

Proof

The middle graph G of Helm graph H_n has $5n + 1$ vertices and $\frac{n(n+23)}{2}$ edges. Based on the eccentric neighbourhood degree sum of vertices of G we partition $V(G)$ into subsets as shown in Table 5 and also we partition $E(G)$ based on the eccentric neighbourhood degree sum of end vertices of edges in G as shown in Table 6. Using formulae (1.1) – (1.3) to this information from Table 5 and Table 6, we obtained the required result.

Acknowledgements:

The authors would like to thank the reviewers for their valuable comments and suggestions to improve the quality of this paper.

References:

[1] H. Wiener, "Structural determination of paraffin boiling points," Journal of the American chemical society, vol. 69, no. 1, pp. 17–20, 1947.
[2] I. Gutman and N. Trinajsti´c, "Graph theory and molecular orbitals. total φ-electron energy of alternant hydrocarbons," Chemical physics letters, vol. 17, no. 4, pp. 535–538, 1972.
[3] C. M. da Fonseca and D. Stevanovic, "Further properties of the second zagreb index," MATCH Commun. Math. Comput. Chem, vol. 72, pp. 655–668, 2014.

[4] I. Gutman and K. C. Das, “The first zagreb index 30 years after,” MATCH Commun. Math. Comput. Chem, vol. 50, no. 1, pp. 83–92, 2004.
[5] M. Khalifeh, H. Yousefi-Azari, and A. R. Ashrafi, “The first and second zagreb indices of some graph operations,” Discrete applied mathematics, vol. 157, no. 4, pp. 804–811, 2009.
[6] V. Sharma, R. Goswami, and A. Madan, “Eccentric connectivity index: A novel highly discriminating topological descriptor for structure- property and structure- activity studies,” Journal of chemical information and computer sciences, vol. 37, no. 2, pp. 273–282, 1997.
[7] H. Dureja and A. K. Madan, “Superaugmented eccentric connectivity indices: new-generation highly discriminating topological descriptors for qsar/qspr modeling,” Medicinal Chemistry Research, vol. 16, pp. 331–341, 2007.
[8] A. Ilic and I. Gutman, “Eccentric connectivity index of chemical trees,” Match-Communications in Mathematical and in Computer Chemistry, no. 65, pp. 731–744, 2011.
[9] S. Sardana and A. Madan, “Application of graph theory: Relationship of molecular connectivity index, wiener’s index and eccentric connectivity index with diuretic activity,” Match-Communications in Mathematical and in Computer Chemistry, no. 43, pp. 85–98, 2001.
[10] M. Morgan, S. Mukwembi, and H. C. Swart, “On the eccentric connectivity index of a graph,” Discrete Mathematics, vol. 311, no. 13, pp. 1229–1234, 2011.
[11] K. Xu, K. Das, S. Klavzar, and H. Li, “Comparison of wiener index and zagreb eccentricity indices,” MATCH-COMMUNICATIONS IN MATHEMATICAL AND IN COMPUTER CHEMISTRY, vol. 84, no. 3, pp. 595–610, 2020.
[12] B. Zhou and Z. Du, “On eccentric connectivity index,” MatchCommunications in Mathematical and in Computer Chemistry, vol. 63, pp. 181–198, 2010.
[13] H. Ahmed, A. Saleh, R. Ismail, A. Alameri, et al., “Computational analysis for eccentric neighborhood zagreb indices and their significance,” Heliyon, vol. 9, no. 7, 2023.
[14] B. Furtula and I. Gutman, “A forgotten topological index,” Journal of mathematical chemistry, vol. 53, no. 4, pp. 1184–1190, 2015.
[15] I. Gutman, B. Furtula, ˇZ. Kovijani´c Vuki´cevi´c, and G. Popivoda, “On zagreb indices and coindices,” MATCH Communications in Mathematical and in Computer Chemistry, 2015.
[16] J. A. Bondy, U. S. R. Murty, et al., Graph theory with applications, vol. 290. Macmillan London, 1976.
[17] M. V. Diudea, I. Gutman, and L. Jantschi, Molecular topology. Nova Science Publishers Huntington, NY, USA, 2001.
[18] I. Gutman and O. E. Polansky, Mathematical concepts in organic chemistry. Springer Science & Business Media, 2012.
[19] N. Trinajstic, Chemical graph theory. CRC press: Boca Raton, Fl, USA, 2nd ed., 1992.
[20] F. Mukhamedov and U. Rozikov, “On inhomogeneous p-adic potts model on a cayley tree,” Infinite Dimensional Analysis, Quantum Probability and Related Topics, vol. 8, no. 02, pp. 277–290, 2005.
[21] G. Mohanapriya and D. Vijayalakshmi, “Eccentricity Indices of wheel graph family”, International Journal of pure and applied mathematics, vol. 117, no. 11, 2017, pp. 17-24.
[22] M. Capobianco and J.C. Molluzzo, “Examples and Counter examples in Graph Theory”, Elesvier Science Ltd, North-Holland (1978).
[23] S. Gupta, M. Singh, A.K. Madan, “Eccentric distance sum: A novel graph invariant for predicting biological and physical properties”, J. Math. Anal. Appl., 275 (2002), 386-401.
[24] Aslam Adnan, Ahmad Safyan, Gao Wei, “On Certain topological indices of Boron triangular nanotube”, Zeitschrift fiir Naturforachung A, 72(8) (2017), 711-716.