Application of Fractional Differential Equations in Electrical Circuits via Sumudu Transform

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Abstract
This paper presents fractional LC, RC and RL electrical circuits as separate cases. The current and charge over time are determined using the Sumudu transform method with emf taken as different types of functions. The results are then simulated using mathematical software.

Keywords: Caputo fractional derivative, Caputo-Fabrizio, Atangana-Baleanu Caputo fractional derivative, Sumudu transform.

1. Introduction

Fractional calculus has become an important area of mathematical research because of its ability to model systems with memory and hereditary characteristics more accurately than classical integer-order calculus. A comprehensive mathematical foundation for fractional differential equations was established by Podlubny [24], while the classical Caputo fractional derivative was introduced by Caputo [25] and has been extensively used in engineering and physical sciences.

The solutions of fractional differential equations are naturally expressed in terms of the Mittag-Leffler function, which serves as the fractional counterpart of the exponential function. The two-parameter Mittag-Leffler function was first introduced by Mittag-Leffler [30], and the one-parameter generalization was studied by Wiman [29]. More recently, Fernandez, Kürt and Ozarslan [17] introduced a naturally emerging bivariate Mittag-Leffler function together with its associated fractional operators, thereby extending the analytical framework of fractional calculus.

To overcome the singularity of classical fractional derivatives, several non-singular operators have been proposed. Caputo and Fabrizio introduced the Caputo–Fabrizio fractional derivative with an exponential kernel [13,14], while Atangana and Baleanu proposed the Atangana–Baleanu fractional derivative with a non-local and non-singular Mittag-Leffler kernel [4]. Abdeljawad [5] further generalized these fractional operators using Mittag-Leffler kernels and investigated their iterated differintegrals. Anderson and Ulness [3] also introduced conformable derivatives, providing another generalized framework for fractional differentiation.

Integral transform techniques play a significant role in obtaining analytical solutions of fractional differential equations. The Laplace transform has been extensively used to solve various classes of fractional differential equations and investigate stability properties [2,32]. The Sumudu transform, introduced by Watugala [28], has become an efficient alternative because it preserves the dimensional properties of functions. Its applications to fractional derivatives and fractional differential equations have been studied by Bodkhe and Panchal [10], Nanaware et al. [21], Nannaware and Patil [22], Jadhav et al. [19,23], and several other researchers. Another useful integral transform, namely the Shehu transform, was applied to Atangana–Baleanu fractional derivatives by Bokhari, Baleanu and Belgacem [9], and later employed in fractional electrical engineering problems by Jadhav et al. [20].

Fractional calculus has found wide-ranging applications in engineering, physics, biology, finance and economics. Atangana and Baleanu [4] demonstrated its effectiveness in heat transfer problems, whereas Bas and Ozarslan [6] discussed several real-world applications of the Atangana–Baleanu derivative. Fractional models have also been developed for COVID-19 transmission by Baleanu, Mohammadi and Rezapour [8], diabetes dynamics by Singh, Kumar and Baleanu [26], and European option pricing by Yavuz and Ozdemir [31].

Economic systems often exhibit memory-dependent behaviour, making fractional calculus an effective mathematical tool for their analysis. Takayama's classical work on mathematical economics [27] and the price adjustment model of Cohen-Vernik and Pazgal [15] provide important foundations for modern economic modelling. Acay, Bas and Abdeljawad [1] formulated fractional economic models based on market equilibrium, while Akgul, Akgul and Baleanu [2] employed the constant proportional Caputo derivative in economic models using the Laplace transform. Further

developments include the application of the Sumudu transform to economic models by Nanaware et al. [21] and analytical solutions of fractional price adjustment equations by Nannaware and Patil [22].
Several researchers have also contributed to the analytical solution of fractional initial value problems using integral transforms. Birajdar, Dole and Goufo [11] investigated analytical solutions of fractional initial value problems, while Debnath and Bhatta [16] provided a comprehensive treatment of integral transforms and their applications. Recent studies by Jadhav et al. [18–20,23] have further demonstrated the effectiveness of Laplace, Sumudu and Shehu transforms in solving fractional differential equations arising in engineering and applied sciences.
Motivated by these developments, the present work focuses on the analytical solution of fractional differential equations involving generalized fractional operators by employing suitable integral transform techniques. The methodology combines the properties of Mittag-Leffler functions with modern fractional derivatives and integral transforms to obtain exact analytical solutions, thereby extending existing results in the literature and providing efficient techniques for solving mathematical, engineering and economic models.

2. Preliminaries

In this section we consider some definitions, theorems, properties and results required in further sections.

Definition 2.1 [29] The classical Mittag-Leffler function with one parameter $E_\alpha(v)$ is

$$E_\alpha(v)=\sum_{k=0}^{\infty}\frac{v^k}{\Gamma(\alpha k+1)},\ \alpha>0$$

Definition 2.2 [30] The classical Mittag-Leffler function with two parameter is $E_{\alpha,\beta}(v)$ is

$$E_{\alpha,\beta}(v)=\sum_{k=0}^{\infty}\frac{v^k}{\Gamma(\alpha k+\beta)},\ v\in c,\beta\in c,\ \ \mathrm{Re}(\alpha)>0$$

Definition 2.3 [29] The generalized Mittag-Leffler function is given by

$$E_{\alpha,\beta}^{\eta}(v)=\sum_{k=0}^{\infty}\frac{v^k(\eta)k}{\Gamma(\alpha k+\beta)k!}\ ,\ v\in c,\ \alpha,\ \beta,\ \eta\in c\ ,\ \mathrm{Re}(\alpha)>0$$

Where $(\eta)k=\eta(\eta+1)....(\eta+k-1)$ is the pochhammer symbol introduced by Prabhakar. Note that $(1)k=k!$ and so $E_{\alpha,\beta}^{1}(v)=E_{\alpha,\beta}(v)$.

Definition 2.4 [30] The Mittag-Leffler function for a special function is given by

$$E_\alpha(\lambda,v)=\sum_{k=0}^{\infty}\frac{\lambda^k v^{\alpha k}}{\Gamma(\alpha k+1)},\ 0\neq\lambda\in R,v\in c,\ \mathrm{Re}(\alpha)>0$$

and

$$E_{\alpha,\beta}(\lambda,v)=\sum_{k=0}^{\infty}\frac{\lambda^k v^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)},\ 0\neq\lambda\in R,v,\beta\in c\ ,\ \mathrm{Re}(\alpha)>0$$

It should be noticed that $E_{\alpha,1}(\lambda,v)=E_\alpha(\lambda,v)$. Also the modified Mittag-Leffler function with three parameters can be written as

$$E_{\alpha,\beta}^{\eta}(v)=\sum_{k=0}^{\infty}\frac{\lambda^k v^{\alpha k+\beta-1}(\eta)k}{\Gamma(\alpha k+\beta)k!},\ 0\neq\lambda\in R,v,\beta\in c\ ,\mathrm{Re}(\alpha)>0$$

Definition 2.5 [12] The left-sided and right-sided Caputo-fractional derivatives of order α are defined by

$${}_a^cD^\alpha v(x)=\frac{1}{\Gamma(1-\alpha)}\int_a^x(x-\tau)^{-\alpha}\ v'(\tau)\ d\tau$$

and

$${}^cD_b^\alpha v(x)=\frac{(-1)}{\Gamma(1-\alpha)}\int_x^b(\tau-x)^{\alpha}\ v'(\tau)\ d\tau\ ,\ 0<\alpha<1$$

Theorem 2.1 [10] The Sumudu transform of Caputo fractional derivative is defined by

$$S\left[{}_0^cD^\alpha v(x)\right](u)=u^{-\alpha}S\left[v(x)\right]-u^{-\alpha}v(0) \tag{2.1}$$

Definition 2.6 Let $\varphi,\psi:[0,\infty)\to R$,then classical convolution product is

$$\left(\phi*\psi\right)(x)=\int_0^x \eta\left(x-s\right)\psi(s)ds$$

Proposition 2.6 Let $\varphi,\psi:\left[0,\infty\right)\to R$, then the following property is valid

$$S\left[\left(\phi*\psi\right)(x)\right]=uS\left[\varphi(x)\right]S\left[\psi(x)\right]$$
$$=u\phi(u)\cdot\psi(u)$$

Definition 2.7 [14] The left-sided and right-sided Caputo-Fabrizio fractional derivatives in the Caputo sense of order α are defined by

$${}_{a}^{CFC}D^{\alpha}w(x)=\frac{M(\alpha)}{\left(1-\alpha\right)}\int_a^x w'(\tau)\ e^{\mu(x-\tau)}d\tau$$

and

$${}^{CFC}D_b^{\alpha}w(x)=-\frac{M(\alpha)}{\left(1-\alpha\right)}\int_x^b w'(\tau)^{\mu(\tau-x)}\ d\tau$$

where $0<\alpha<1$, $M(\alpha)$ is normalization function and $\mu=-\dfrac{\alpha}{1-\alpha}$

Theorem 2.2 [8] The Sumudu transform of CFC fractional derivative $w(x)$ is

$$S\left[{}_{0}^{CFC}D_x^{\alpha}w(x)\right]=\frac{M(\alpha)}{\left(1-\alpha+\alpha u\right)}\left[S\left[w(x)\right]-w(0)\right] \tag{2.2}$$

Definition 2.8 [4] The left-sided and right-sided Atangana-Baleanu fractional derivatives in the Caputo sense of order α are defined by

$${}_{a}^{ABC}D^{\alpha}w(x)=\frac{B(\alpha)}{\left(1-\alpha\right)}\int_a^x w'(\tau)\ \ E_{\alpha}\left[\mu\left(x-\tau\right)\right]d\tau$$

and

$${}^{ABC}D_b^{\alpha}w(x)=-\frac{B(\alpha)}{\left(1-\alpha\right)}\int_x^b w'(\tau)\ \ E_{\alpha}\left[\mu\left(\tau-x\right)\right]d\tau$$

where $0<\alpha<1, B(\alpha)$ is normalization function and $\mu=-\dfrac{\alpha}{1-\alpha}$

Theorem 2.3 [9] The Sumudu transform of ABC fractional derivative is defined by

$$S\left[{}_{0}^{ABC}D^{\alpha}w(x)\right]=\frac{B(\alpha)}{\left(1-\alpha+\alpha u^{\alpha}\right)}\left[S\left[w(x)-w(0)\right]\right] \tag{2.3}$$

Lemma 2.1 The Sumudu transform of certain functions holds:

i. $$S\left[E_{\alpha}\left(-ax^{\alpha}\right)\right]=\frac{1}{1+au^{\alpha}}$$

ii. $$S\left[1-E_{\alpha}\left(-ax^{\alpha}\right)\right]=\frac{au^{\alpha}}{1+au^{\alpha}}$$

iii. $$S\left[x^{\alpha-1}E_{\alpha,\alpha}\left(-ax^{\alpha}\right)\right]=\frac{u^{\alpha-1}}{1+au^{\alpha}}$$

Proof. Since

$$\sum_{p=0}^{\infty}\frac{\left(p+m\right)!}{p!}x^{p}=\frac{m!}{\left(1-x\right)^{(m+1)}}$$

i. $$S\left[E_{\alpha}\left(-ax^{\alpha}\right)\right]=\sum_{p=0}^{\infty}\frac{\left(p+m\right)!}{p!}(-a)^{p}\frac{S\left[x^{\alpha p}\right]}{\Gamma\left(\alpha p+1\right)}$$
$$=\sum_{p=0}^{\infty}\frac{\left(p+m\right)!}{p!}(-a)^{p}\frac{\Gamma\left(\alpha p+1\right)}{\Gamma\left(\alpha p+1\right)}u^{\alpha p}$$

$$= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} (-a)^p \; u^{\alpha p}$$

$$\therefore \quad S\left[E_\alpha\left(-ax^\alpha\right)\right] = \frac{1}{1+au^\alpha} \qquad (\because m=0)$$

ii.

$$S\left[1-E_\alpha\left(-ax^\alpha\right)\right] = 1-\frac{1}{1+au^\alpha}$$

$$= \frac{au^\alpha}{1+au^\alpha}$$

iii.

$$S\left[x^{\alpha-1}E_{\alpha,\alpha}\left(-ax^\alpha\right)\right] = \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} (-a)^p \frac{S\left[x^{\alpha p+\alpha-1}\right]}{\Gamma(\alpha p+\alpha)}$$

$$= \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} (-a)^p \frac{\Gamma(\alpha p+\alpha)}{\Gamma(\alpha p+\alpha)} \; u^{\alpha p+\alpha-1}$$

$$= u^{\alpha-1} \sum_{p=0}^{\infty} \frac{(p+m)!}{p!} (-a)^p \; u^{\alpha p}$$

$$\therefore \quad S\left[x^{\alpha-1}E_{\alpha,\alpha}\left(-ax^\alpha\right)\right] = \frac{u^{\alpha-1}}{1+au^\alpha} \qquad (\because m=0)$$

3. RC Electrical circuit

Let an electric circuit containing a resistance(R) and a inductance(L) are positive constants and E is the applied electro motive force with zero. According to the Kirchoff's Voltage laws, the equation of the RC series circuit is given by

The voltage drop across the resistance(R)

$$V_R(t) = Ri(t)$$

The voltage drop across the capacitance(C)

$$V_C(t) = \frac{1}{C}\int_0^t i(x)dx = \frac{q(t)}{C}$$

The ordinary differential equation for RC circuit is

$$Rq'(t) + \frac{q(t)}{C} = E(t)$$

The fractional form of above differential equation is

$$D_t^\alpha q(t) + \frac{q(t)}{RC} = \frac{E(t)}{R} \tag{3.1}$$

The fractional equation in ABC sense for the electrical circuit RC is given by

$${}^{ABC}D_t^\alpha q(t) = \frac{E(t)}{R} - \frac{q(t)}{RC}$$

where $\mu = \dfrac{\lambda^{1-\alpha}}{RC} = \mu_0 \lambda^{1-\alpha}$, μ is the fractional time constant and $\mu_0 = \dfrac{1}{RC}$ is the time constant in the classical case. Now we obtain the analytical solutions of equation (3.1) considering different source terms.

CASE-I: Unit step source $E(t) = p(t)V$; $q(0) = q_0(t)V, \; (q_0 > 0)$

Equation (3.1) can be written as

$${}^{ABC}D_t^\alpha q(t) = \mu V p(t) - \mu q(t)$$

Applying Sumudu transform, we have

$$S\left[{}^{ABC}D_t^\alpha q(t)\right] = \mu V - \mu S\left[q(t)\right]$$

$$\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\left[S\left[q(t)\right]-q(0)\right]=\mu V-\mu S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}+\mu\right]=\left[\mu V+\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)+\mu\left(1-\alpha+\alpha u^{\alpha}\right)}{1-\alpha+\alpha u^{\alpha}}\right]=\left[\frac{V\mu\left(1-\alpha+\alpha u^{\alpha}\right)+B(\alpha)q_0}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]=\frac{\mu V\left(1-\alpha+\alpha u^{\alpha}\right)}{B(\alpha)+\mu(1-\alpha+\alpha u^{\alpha})}+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha+\alpha u^{\alpha})}$$

$$=\frac{\mu V}{B(\alpha)+\mu(1-\alpha+\alpha u^{\alpha})}\left[\frac{1-\alpha}{1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}}+\frac{\alpha u^{\alpha}}{1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}}\right]+$$

$$\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}\left[\frac{1}{1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}}\right]$$

Applying inverse Sumudu transform to the above equation yields

$$q(t)=\frac{\mu V}{B(\alpha)+\mu(1-\alpha)}S^{-1}\left[\frac{1-\alpha}{1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}}+\frac{\alpha u^{\alpha-1}u^{\alpha}}{1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}S^{-1}\left[\frac{1}{1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}}\right]$$

$$q(t)=\frac{\mu(1-\alpha)V}{B(\alpha)+\mu(1-\alpha)}E_{\alpha,1}\left[-\left(\frac{\mu\alpha}{B(\alpha)+\mu(1-\alpha)}\right)t^{\alpha}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}E_{\alpha,1}\left[-\left(\frac{\mu\alpha}{B(\alpha)+\mu(1-\alpha)}\right)t^{\alpha}\right]$$

$$+\frac{\mu(\alpha)V}{B(\alpha)+\mu(1-\alpha)}\int_0^t\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left[-\left(\frac{\mu\alpha}{B(\alpha)+\mu(1-\alpha)}\right)\right]\left(t-\tau\right)^{\alpha}d\tau$$

Case II :- Exponential source $E(t)=e^{-at}V$; $q(0)=q_0(t)V,\ (q_0>0)$ equation (3.1) can be written as

$${}^{ABC}D_t^{\alpha}q(t)=\mu Ve^{-at}-\mu q(t)$$

Applying the Sumudu transform, we have

$$S\left[{}^{ABC}D_t^{\alpha}q(t)\right]=\mu VS\left[e^{-at}\right]-\mu S\left[q(t)\right]$$

$$\frac{B(\alpha)}{(1-\alpha)+\alpha u^{\alpha}}\left[S\left[q(t)\right]-q(0)\right]=\mu V\frac{1}{1+au}-\mu S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)}{(1-\alpha)+\alpha u^{\alpha}}+\mu\right]=\left[\mu V\frac{1}{1+au}+\frac{B(\alpha)}{(1-\alpha)+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)+\mu\left((1-\alpha)+\alpha u^{\alpha}\right)}{(1-\alpha)+\alpha u^{\alpha}}\right]=\left[\frac{\frac{\mu V}{1+au}\left((1-\alpha)+\alpha u^{\alpha}\right)+B(\alpha)q_0}{(1-\alpha)+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]=\frac{\mu V\left((1-\alpha)+\alpha u^{\alpha}\right)}{\left(1+au\right)\left(B\left(\alpha\right)+\mu\left((1-\alpha)+\alpha u^{\alpha}\right)\right)}+\frac{B(\alpha)q_0}{\left(B\left(\alpha\right)+\mu\left((1-\alpha)+\alpha u^{\alpha}\right)\right)}$$

$$=\frac{\mu V}{B(\alpha)+\mu(1-\alpha)}\left[\frac{1-\alpha}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{1}{\left(1+au\right)}+\frac{\alpha u^{\alpha}}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{1}{\left(1+au\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}\left[\frac{1}{1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}}\right]$$

Applying inverse Sumudu transform to the above equation yields

$$q(t)=\frac{\mu V}{B(\alpha)+\mu(1-\alpha)}S^{-1}\left[\frac{1-\alpha}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{1}{\left(1+au\right)}+\frac{\alpha u^{\alpha}}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{1}{\left(1+au\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}S^{-1}\left[\frac{1}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\right]$$

$$q(t)=\frac{\mu(1-\alpha)V}{B(\alpha)+\mu(1-\alpha)}\int_0^t\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left[-\left(\frac{\mu\alpha}{B(\alpha)+\mu(1-\alpha)}\right)\left(t-\tau\right)^{\alpha}\right]d\tau$$

$$\cdot\tau^{-\alpha}E_{\alpha,1-\alpha}\left(-a\tau\right)d\tau+\frac{\mu\alpha}{B(\alpha)+\mu(1-\alpha)}\int_0^t\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left[-\left(\frac{\mu\alpha}{B(\alpha)+\mu(1-\alpha)}\right)\left(t-\tau\right)^{\alpha}\right]d\tau$$

$$\cdot E_{1,1}\left(-a\mu\right)d\mu+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}E_{\alpha,1}\left(-\left[-\left(\frac{\mu\alpha}{B(\alpha)+\mu(1-\alpha)}\right)\left(t-\tau\right)^{\alpha}\right]\right)$$

Case III: Periodic source $E(t)=\sin atV$; $q(0)=q_0V,(q_0>0)$ equation (3.1) can be written as,

$${}^{ABC}D_t^{\alpha}q(t)=\mu V\sin at-\mu q(t)$$

Applying the Sumudu transform, we have

$$S\left[{}^{ABC}D_t^{\alpha}q(t)\right]=\mu VS\left[\sin at\right]-\mu S\left[q(t)\right]$$

$$\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\left[S\left[q(t)\right]-q(0)\right]=\mu V\frac{au}{1+a^2u^2}-\mu S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}+\mu\right]=\left[\mu V\frac{au}{1+a^2u^2}+\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)+\mu\left(1-\alpha+\alpha u^{\alpha}\right)}{1-\alpha+\alpha u^{\alpha}}\right]=\left[\frac{V\mu\frac{au}{1+a^2u^2}\left(1-\alpha+\alpha u^{\alpha}\right)+B(\alpha)q_0}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]=\frac{au\mu V\left(1-\alpha+\alpha u^{\alpha}\right)}{\left(1+a^2u^2\right)B(\alpha)+\mu\left(1-\alpha+\alpha u^{\alpha}\right)}+\frac{B(\alpha)q_0}{B(\alpha)+\mu\left(1-\alpha+\alpha u^{\alpha}\right)}$$

$$=\frac{\mu V}{B(\alpha)+\mu(1-\alpha)}\left[\frac{1-\alpha}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{au}{\left(1+a^2u^2\right)}+\frac{\alpha u^{\alpha}}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{au}{\left(1+a^2u^2\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}\left[\frac{1}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\right]$$

Applying inverse Sumudu transform to the above equation yields

$$q(t)=\frac{\mu V}{B(\alpha)+\mu(1-\alpha)}S^{-1}\left[\frac{1-\alpha}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{au}{\left(1+a^2u^2\right)}+\frac{\alpha u^{\alpha}}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\frac{au}{\left(1+a^2u^2\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}S^{-1}\left[\frac{1}{\left(1+\frac{\mu\alpha u^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)}\right]$$

$$q(t)=\frac{\mu(1-\alpha)^2V}{B(\alpha)+\mu(1-\alpha)}\int_0^t\frac{d}{dx}\int_0^x\delta(a)E_{\alpha,1}\left[-\left(\frac{\mu\alpha(x-a)}{B(\alpha)+\mu(1-\alpha)}\right)\right]da$$

$$\cdot\sin\left(a(t-x)\right)dx+\frac{B(\alpha)q_0}{B(\alpha)+\mu(1-\alpha)}E_{\alpha,1}\left[-\left(\frac{(\mu t)^{\alpha}}{B(\alpha)+\mu(1-\alpha)}\right)\right]$$

4 LC electrical circuit

Let an electric circuit containing a inductance (L) and a capacitance (C) are positive constants and E is the applied electro motive force with zero. According to the Kirchoff's Voltage laws, the equation of the LR series circuit is given by.

The voltage drop across the capacitance(C)

$V_R(t)=Ri(t)$

The voltage drop across the capacitance(C)

$V_C(t)=\frac{1}{C}\int_0^t i(x)dx=\frac{q(t)}{C}$

The ordinary differential equation for LC circuit is

$Lq''(t)+\frac{q}{C}=E(t)$

The fractional form of above differential equation is

$$D_t^{2\alpha}q(t)+\frac{q(t)}{LC}=\frac{1}{L}E(t)$$

The fractional equation in ABC sense for the electrical circuit LC is given by

$$^{ABC}D_t^{\alpha}q(t)=k^2CE(t)-k^2q(t) \tag{4.1}$$

Where,

$k^2=\dfrac{\lambda^{1-\alpha}}{LC}=k_0^2\lambda^{2(1-\alpha)}$ is the fractional angular frequency and $k_0=\dfrac{1}{\sqrt{LC}}$ is the natural frequency in the classical.

Now we obtain the analytical solutions of equation (4.1) considering different source terms.

Case I :- Unit step source $E(t)=p(t)V$; $q(0)=q_0A,\ (q_0>0)$ equation (4.1) can be written as

$$^{ABC}D_t^{\alpha}q(t)=k^2Cp(t)-k^2q(t)$$

Applying the Sumudu transform, we have

$$S\left[{}^{ABC}D_t^{2\alpha}q(t)\right]=k^2Cp(t)-k^2S\left[q(t)\right]$$

$$\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}\left[S\left[q(t)\right]-q(0)\right]=k^2Cp(t)-k^2S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}+k^2\right]=k^2C-\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}q(0)$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}\right]=\frac{k^2C\left(1-\alpha-\alpha u^{\alpha}\right)^2+B(\alpha)^2q_0}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}$$

$$S\left[q(t)\right]=\frac{k^2C\left(1-\alpha-\alpha u^{\alpha}\right)^2}{B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2}+\frac{B(\alpha)^2q_0}{B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2}$$

$$S\left[q(t)\right]=\frac{k^2C}{\left[B(\alpha)+ik\left(1-\alpha\right)\right]\left[B(\alpha)-ik\left(1-\alpha\right)\right]}\left[\frac{\left(1-\alpha\right)^2+2\alpha(1-\alpha)u^{\alpha}+\left(\alpha u^{\alpha}\right)^2}{\left(1+\dfrac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)\left(1-\dfrac{ik\alpha u^{\alpha}}{B(\alpha)-ik\left(1-\alpha\right)}\right)}\right]$$

$$+\frac{B(\alpha)^2q_0}{\left[B(\alpha)+ik\left(1-\alpha\right)\right]\left[B(\alpha)-ik\left(1-\alpha\right)\right]}\left[\frac{1}{\left(1+\dfrac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)\left(1-\dfrac{ik\alpha u^{\alpha}}{B(\alpha)-ik\left(1-\alpha\right)}\right)}\right]$$

Applying inverse Sumudu transform to the above equation yields

$$q(t)=\frac{k^2C\left(1-\alpha\right)^2}{B(\alpha)^2+k^2\left(1-\alpha\right)^2}\int_0^t\left(t-\tau\right)^{-\alpha-1}E_{\alpha,-\alpha}\left[-\left(\frac{ik\alpha}{B(\alpha)+ik\left(1-\alpha\right)}\right)\left(t-\tau\right)^{\alpha}\right]$$

$$\cdot\tau^{\alpha}E_{\alpha,\alpha+1}\left[\left(\frac{ik\alpha}{B(\alpha)-ik\left(1-\alpha\right)}\right)t^{\alpha}\right]d\tau$$

$$+\frac{2k^2C\left(1-\alpha\right)^2}{B(\alpha)^2+k^2\left(1-\alpha\right)^2}\int_0^t E_{\alpha,1}\left[-\left(\frac{ik\alpha}{B(\alpha)+ik\left(1-\alpha\right)}\right)\left(t-\tau\right)^{\alpha}\right]$$

$$\cdot\tau^{\alpha-1}E_{\alpha,\alpha}\left[\left(\frac{ik\alpha}{B(\alpha)-ik(1-\alpha)}\right)t^{\alpha}\right]d\tau$$

$$+\frac{k^2C\alpha^2}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t E_{\alpha,\alpha+1}\left[-\left(\frac{ik\alpha}{B(\alpha)+ik(1-\alpha)}\right)(t-\tau)^{\alpha}\right]$$

$$\cdot\tau^{\alpha-1}E_{\alpha,\alpha}\left[\left(\frac{ik\alpha}{B(\alpha)-ik(1-\alpha)}\right)t^{\alpha}\right]d\tau$$

$$+\frac{B(\alpha)^2q_0}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,-\alpha}\left[-\left(\frac{ik\alpha}{B(\alpha)+ik(1-\alpha)}\right)(t-\tau)^{\alpha}\right]$$

$$\cdot\tau^{\alpha}E_{\alpha,\alpha+1}\left[\left(\frac{ik\alpha}{B(\alpha)-ik(1-\alpha)}\right)t^{\alpha}\right]d\tau$$

Case II :- Exponential source $E(t)=e^{-at}V$; $q(0)=q_0A,\ (q_0>0)$ equation (4.1) can be written as,

$$^{ABC}D_t^{\alpha}q(t)=k^2Ce^{-at}-k^2q(t)$$

Applying the Sumudu transform, we have,

$$S\left[{}^{ABC}D_t^{2\alpha}q(t)\right]=k^2C\frac{1}{(1+au)}-k^2S\left[q(t)\right]$$

$$\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}\left[S\left[q(t)\right]-q(0)\right]=k^2C\frac{1}{(1+au)}-k^2S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}+k^2\right]=k^2C\frac{1}{(1+au)}-\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}q(0)$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}\right]=\frac{k^2C\left(1-\alpha-\alpha u^{\alpha}\right)^2\frac{1}{(1+au)}+B(\alpha)^2q_0}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}$$

$$S\left[q(t)\right]=\frac{k^2C\left(1-\alpha-\alpha u^{\alpha}\right)^2}{(1+au)\left(B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2\right)}+\frac{B(\alpha)^2q_0}{B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2}$$

The expression for the current is,

$$q(t)=\frac{k^2C(1-\alpha)^2}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t E_{\alpha,1}\left[-\left(\frac{ik\alpha}{B(\alpha)+ik(1-\alpha)}\right)(t-\tau)^{\alpha}\right]$$

$$\cdot\tau^{-\alpha}E_{\alpha,-2}\left[\left(\frac{ik\alpha}{B(\alpha)+ik(1-\alpha)}\right)t^{\alpha}\right]d\tau*\left(\frac{u}{1+au}\right)$$

$$+\frac{2k^2C\alpha(1-\alpha)}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t E_{\alpha,1}\left[-\left(\frac{ik\alpha}{B(\alpha)+ik(1-\alpha)}\right)(\tau)^{\alpha}\right]$$

$$\cdot\tau^{\alpha-1}E_{\alpha,\alpha}\left[\left(\frac{ik\alpha}{B(\alpha)-ik(1-\alpha)}\right)\tau^{\alpha}\right]d\tau*\left(\frac{u}{1+au}\right)$$

$$+\frac{B(\alpha)^2q_0}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,-\alpha}\left[-\left(\frac{ik\alpha}{B(\alpha)+ik(1-\alpha)}\right)(t-\tau)^{\alpha}\right]$$

$$\cdot\tau^{\alpha}E_{\alpha,\alpha+1}\left[\left(\frac{ik\alpha}{B(\alpha)+ik\left(1-\alpha\right)}\right)\tau^{\alpha}\right]d\tau$$

Applying inverse Sumudu transform to the above equation yields,

$$q(t)=\frac{k^2C\left(1-\alpha\right)^2}{B(\alpha)^2+k^2\left(1-\alpha\right)^2}\int_0^t\frac{mE_{\alpha,-1}\left(m\tau^{\alpha}\right)+nE_{\alpha,-1}\left(-n\tau^{\alpha}\right)}{m+n}\tau^{-2}$$

$$\cdot E_{1,1}\left[\left(-m\left(t-\tau\right)\right)-1\right]d\tau$$

$$+\frac{2k^2C\left(1-\alpha\right)^2}{B(\alpha)^2+k^2\left(1-\alpha\right)^2}\int_0^t\frac{mE_{\alpha,\alpha}\left(m\tau^{\alpha}\right)+nE_{\alpha,\alpha}\left(-n\tau^{\alpha}\right)}{m+n}\tau$$

$$\cdot E_{1,1}\left[\left(-m\left(t-\tau\right)\right)\right]$$

$$+\frac{k^2C\alpha^2}{B(\alpha)^2+k^2\left(1-\alpha\right)^2}\int_0^t\frac{mE_{\alpha,2\alpha}\left(m\tau^{\alpha}\right)+nE_{\alpha,2\alpha}\left(-n\tau^{\alpha}\right)}{m+n}\tau^{2\alpha-1}$$

$$\cdot E_{1,1}\left[\left(-m\left(t-\tau\right)\right)\right]$$

$$+\frac{B(\alpha)^2q_0}{B(\alpha)^2+k^2\left(1-\alpha\right)^2}\left[\frac{mE_{\alpha,-1}\left(m\tau^{\alpha}\right)+nE_{\alpha,-1}\left(-n\tau^{\alpha}\right)}{m+n}\right]$$

Where,

$$m=\frac{ik\alpha}{B(\alpha)-ik\left(1-\alpha\right)}\ \text{ and }\ n=\frac{ik\alpha}{B(\alpha)+ik\left(1-\alpha\right)}$$

Case III :- Periodic source $E(t)=\sin atV$; $q(0)=q_0A,\ (q_0>0)$ equation (4.1) can be written as,

$${}^{ABC}D_t^{\alpha}q(t)=k^2C\sin at-k^2q(t)$$

Applying the Sumudu transform, we have

$$S\left[{}^{ABC}D_t^{2\alpha}q(t)\right]=k^2CS\left[\sin at\right]-k^2S\left[q(t)\right]$$

$$\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}\left[S\left[q(t)\right]-q(0)\right]=k^2C\frac{au}{\left(1+a^2u^2\right)}-k^2S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}+k^2\right]=k^2C\frac{au}{\left(1+a^2u^2\right)}-\frac{B(\alpha)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}q(0)$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}\right]=\frac{k^2C\left(1-\alpha-\alpha u^{\alpha}\right)^2\frac{au}{\left(1+a^2u^2\right)}+B(\alpha)^2q_0}{\left(1-\alpha-\alpha u^{\alpha}\right)^2}$$

$$S\left[q(t)\right]=\frac{auk^2C\left(1-\alpha-\alpha u^{\alpha}\right)^2}{\left(1+a^2u^2\right)\left(B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2\right)}+\frac{B(\alpha)^2q_0}{B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2}$$

$$S\left[q(t)\right]=\frac{k^2C}{\left(B(\alpha)^2+k^2\left(1-\alpha-\alpha u^{\alpha}\right)^2\right)}\left[\frac{\left(1-\alpha\right)^2}{\left(1+\frac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)\left(1+\frac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)}\frac{au}{\left(1+a^2u^2\right)}+\frac{2\alpha(1-\alpha)u^{\alpha}}{\left(1+\frac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)\left(1+\frac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)}\frac{au}{\left(1+a^2u^2\right)}+\frac{(\alpha u^{\alpha})^2}{\left(1+\frac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)\left(1+\frac{ik\alpha u^{\alpha}}{B(\alpha)+ik\left(1-\alpha\right)}\right)}\frac{au}{\left(1+a^2u^2\right)}\right]$$

$$+\frac{B(\alpha)^2 q_0}{B(\alpha)^2+k^2(1-\alpha)^2}\left[\frac{1}{\left(1+\frac{ik\alpha u^\alpha}{B(\alpha)+ik(1-\alpha)}\right)\left(1+\frac{ik\alpha u^\alpha}{B(\alpha)+ik(1-\alpha)}\right)}\right]$$

Applying inverse Sumudu transform to the above equation yields

$$q(t)=\frac{k^2C(1-\alpha)^2}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t \frac{mE_{\alpha,1}\left(n\tau^\alpha\right)+nE_{\alpha,-1}\left(-m\tau^\alpha\right)}{m+n}\tau^{-2}\left[\frac{1-\cos\left(m(t-\tau)\right)}{m}\right]d\tau$$

$$+\frac{2k^2C\alpha(1-\alpha)}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t \frac{mE_{\alpha,\alpha}\left(m\tau^\alpha\right)+nE_{\alpha,\alpha}\left(-n\tau^\alpha\right)}{m+n}\tau\sin\left(m(t-\tau)\right)d\tau$$

$$+\frac{k^2C\alpha^2}{B(\alpha)^2+k^2(1-\alpha)^2}\int_0^t \frac{mE_{\alpha,2\alpha}\left(m\tau^\alpha\right)+nE_{\alpha,2\alpha}\left(-n\tau^\alpha\right)}{m+n}\tau^{2\alpha-1}\sin\left(m(t-\tau)\right)d\tau$$

$$+\frac{B(\alpha)^2 q_0}{B(\alpha)^2+k^2(1-\alpha)^2}\left[\frac{mE_{\alpha,1}\left(n\tau^\alpha\right)+nE_{\alpha,-1}\left(-m\tau^\alpha\right)}{m+n}\right]$$

Where,

$$m=\frac{ik\alpha}{B(\alpha)-ik(1-\alpha)} \text{ and } n=\frac{ik\alpha}{B(\alpha)+ik(1-\alpha)}$$

5 LR electrical circuit

Let an electric circuit containing a resistance (R) and a inductance (L) are positive constants and E is the applied electro motive force with E(t). According to the Kirchoff's Voltage laws, the equation of the LR series circuit is given by,

The voltage drop across the resistance(R)

$$V_R(t)=Ri(t)$$

The voltage drop across the inductance(L)

$$V_L(t)=L\frac{di}{dt}$$

The ordinary differential equation for RC circuit is

$$Lq'(t)+Rq(t)=E(t)$$

The fractional form of above differential equation is,

$$D_t^\alpha i(t)+\frac{R}{L}q(t)=\frac{1}{L}E(t)$$

The fractional equation in ABC sense for the electrical circuit RL is given by,

$${}^{ABC}D_t^\alpha q(t)=\frac{\omega}{R}E(t)-\omega q(t)$$

Where, $\omega=\frac{\lambda^{1-\alpha}R}{L}=\omega_0\lambda^{1-\alpha}\omega$, is the fractional time constant and $\omega_0=LR$ is the time constant in the classical case.

Now, we obtain the analytical solutions of equation (5.1) considering different source terms.

Case I :- Unit step source $E(t)=p(t)V$; $q(0)=q_0V,\ (q_0>0)$ equation (3.1) can be written as

$${}^{ABC}D_t^\alpha q(t)=\frac{1}{R}\omega P(t)-\omega q(t)$$

Applying the Sumudu transform, we have ,

$$S\left[{}^{ABC}D_t^\alpha q(t)\right]=\frac{1}{R}\omega P(t)-\omega S\left[q(t)\right]$$

$$\frac{B(\alpha)}{1-\alpha+\alpha u^\alpha}\left[S\left[q(t)\right]-q(0)\right]=\frac{1}{R}\omega-\omega S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}+\omega\right]=\left[\frac{1}{R}\omega+\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)}{1-\alpha+\alpha u^{\alpha}}\right]=\left[\frac{\frac{1}{R}\omega\left(1-\alpha+\alpha u^{\alpha}\right)+B(\alpha)q_0}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]=\frac{\omega\left(1-\alpha+\alpha u^{\alpha}\right)}{R\left[B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)\right]}+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)}$$

$$=\frac{\omega}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}\left[\frac{1-\alpha}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}+\frac{\alpha u^{\alpha}}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}\left[\frac{1}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

Applying inverse Sumudu transform to the above equation yields

$$q(t)=\frac{\omega}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}S^{-1}\left[\frac{1-\alpha}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}+\frac{\alpha u^{\alpha}}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}S^{-1}\left[\frac{1}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

$$q(t)=\frac{\omega\left(1-\alpha\right)}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}E_{\alpha,1}\left[-\left(\frac{\omega\alpha}{B(\alpha)+\omega\left(1-\alpha\right)}\right)t^{\alpha}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}E_{\alpha,1}\left[-\left(\frac{\omega\alpha}{B(\alpha)+\omega\left(1-\alpha\right)}\right)t^{\alpha}\right]$$

$$+\frac{\omega(\alpha)}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}\int_0^t\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left[-\left(\frac{\omega\alpha}{B(\alpha)+\omega\left(1-\alpha\right)}\right)\left(t-\tau\right)^{\alpha}\right]d\tau$$

Case II :- Exponential source $E(t)=e^{-at}V$; $q(0)=q_0V$, $\left(q_0>0\right)$ equation (5.1) can be written as

$$^{ABC}D_t^{\alpha}q(t)=\frac{1}{R}\omega e^{-at}-\omega q(t)$$

Applying the Sumudu transform, we have

$$S\left[{}^{ABC}D_t^{\alpha}q(t)\right]=\frac{1}{R}\omega S\left[e^{-at}\right]-\omega S\left[q(t)\right]$$

$$\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\left[S\left[q(t)\right]-q(0)\right]=\frac{\omega}{R\left(1+au\right)}-\omega S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}+\omega\right]=\left[\frac{\omega}{R(1+au)}+\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)}{1-\alpha+\alpha u^{\alpha}}\right]=\left[\frac{\frac{\omega}{R(1+au)}\left(1-\alpha+\alpha u^{\alpha}\right)+B(\alpha)q_0}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]=\frac{\omega\left(1-\alpha+\alpha u^{\alpha}\right)}{R\left(1+au\right)\left[B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)\right]}+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)}$$

$$=\frac{\omega}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}\left[\frac{1-\alpha}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{1}{\left(1+au\right)}+\frac{\alpha u^{\alpha}}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{1}{\left(1+au\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}\left[\frac{1}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

Applying inverse Sumudu transform to the above equation yields,

$$q(t)=\frac{\omega}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}S^{-1}\left[\frac{1-\alpha}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{1}{\left(1+au\right)}+\frac{\alpha u^{\alpha}}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{1}{\left(1+au\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}S^{-1}\left[\frac{1}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

$$q(t)=\frac{\omega\left(1-\alpha\right)}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}\int_0^t\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left[-\left(\frac{\omega\alpha}{B(\alpha)+\omega\left(1-\alpha\right)}\right)(t-\tau)^{\alpha}\right]\tau^{-\alpha}E_{1,1-\alpha}\left(-a\tau\right)d\tau$$

$$+\frac{\omega\alpha}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}\int_0^t\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left[-\left(\frac{\omega\alpha}{B(\alpha)+\omega\left(1-\alpha\right)}\right)(t-\tau)^{\alpha}\right]E_{1,1-\alpha}\left(-a\tau\right)d\tau$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}E_{\alpha,1}\left[-\left(\frac{\omega\alpha}{B(\alpha)+\omega\left(1-\alpha\right)}\right)t^{\alpha}\right]$$

Case III :- Periodic source $E(t)=\sin atV$; $q(0)=q_0V,\ \left(q_0>0\right)$ equation (5.1) can be written as

$${}^{ABC}D_t^{\alpha}q(t)=\frac{1}{R}\omega\sin at-\omega q(t)$$

Applying the Sumudu transform, we have,

$$S\left[{}^{ABC}D_t^{\alpha}q(t)\right]=\frac{1}{R}\omega S\left[\sin at\right]-\omega S\left[q(t)\right]$$

$$\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\left[S\left[q(t)\right]-q(0)\right]=\frac{au\omega}{R\left(1+a^2u^2\right)}-\omega S\left[q(t)\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}+\omega\right]=\left[\frac{au\omega}{R(1+a^2u^2)}+\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]\left[\frac{B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)}{1-\alpha+\alpha u^{\alpha}}\right]=\left[\frac{\frac{au\omega}{R(1+a^2u^2)}\left(1-\alpha+\alpha u^{\alpha}\right)+B(\alpha)q_0}{1-\alpha+\alpha u^{\alpha}}\right]$$

$$S\left[q(t)\right]=\frac{au\omega\left(1-\alpha+\alpha u^{\alpha}\right)}{R\left(1+a^2u^2\right)\left[B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)\right]}+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha+\alpha u^{\alpha}\right)}$$

$$=\frac{\omega}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}\left[\frac{1-\alpha}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{au}{\left(1+a^2u^2\right)}+\frac{\alpha u^{\alpha}}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{au}{\left(1+a^2u^2\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}\left[\frac{1}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

Applying inverse Sumudu transform to the above equation yields,

$$q(t)=\frac{\omega}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}S^{-1}\left[\frac{1-\alpha}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{au}{\left(1+a^2u^2\right)}+\frac{\alpha u^{\alpha}}{\left[1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right]}\frac{au}{\left(1+a^2u^2\right)}\right]$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}S^{-1}\left[\frac{1}{1+\frac{\omega\alpha u^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}}\right]$$

$$q(t)=\frac{\omega\left(1-\alpha\right)^2}{R\left[B(\alpha)+\omega\left(1-\alpha\right)\right]}\int_0^t\frac{d}{dx}\int_0^x\delta(a)E_{\alpha,1}\left[-\left(\frac{\omega\alpha(x-a)}{B(\alpha)+\omega\left(1-\alpha\right)}\right)\right]da\sin\left(a\left(t-x\right)\right)dx$$

$$+\frac{B(\alpha)q_0}{B(\alpha)+\omega\left(1-\alpha\right)}E_{\alpha,1}\left[-\left(\frac{(\omega t)^{\alpha}}{B(\alpha)+\omega\left(1-\alpha\right)}\right)\right]$$

6 Conclusion

In this paper, the Sumudu transform method has been successfully applied to obtain analytical solutions of fractional economic models involving the constant proportional Caputo fractional derivative. The proposed approach provides a simple, efficient, and systematic procedure for solving fractional differential equations while preserving the physical dimensions of the original problem. The solutions are expressed in terms of generalized Mittag-Leffler functions and satisfy the prescribed initial conditions, demonstrating the effectiveness of the transform technique.

The results show that fractional-order models provide a more realistic representation of economic systems by incorporating memory and hereditary effects that are not captured by classical integer-order models. The combination of the constant proportional Caputo derivative and the Sumudu transform offers an efficient mathematical framework for analyzing the dynamic behavior of economic variables and can be extended to various fractional models arising in economics, finance, engineering, and other applied sciences.

Future research may focus on extending the proposed methodology to nonlinear, variable-order, and multi-dimensional fractional economic models. In addition, numerical simulations, stability analysis, and validation using real economic data may further demonstrate the applicability and predictive capability of the proposed fractional modeling approach.

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