Viscous oscillatory exponentially stratified flow through Rectangular Narrow Channels

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Abstract
The research work in this article proposes a mathematical model and its results to the flow which is oscillatory and vertically exponential density stratified viscous fluid induced by constant suction through one side of the channel. The region of the flow is assumed to be narrow rectangular shaped formed by two parallel plates of which the right-hand side plate is porous and the fluid is taken out through this plate inducing the flow with constant velocity. The mathematical system explaining the above problem is formed using narrow channel approximation and solved using variable separable similarity solution and were interpreted through graphs for various parameter involved in the system. The stratification is considered to be exponentially distributed in the vertical direction and the results reveal that the flow in the vertical direction is mainly affected by the stratification parameter and significant comparisons are established with the case of linearly distributed density stratification. Further for second-degree approximation, the exponential stratification reduces to the results of linear stratification results, which are already available in the literature.

Key Words: Environmental flows, Density stratification, Oscillatory flow, Exponential stratification, Similarity transformation, Narrow channel approximation.

Introduction

Stratification is a phenomenon that occurs in nature such as in Sea Water, Rivers and in Lakes. There are various types of stratification that exists in nature. Most important of them are thermal stratification, viscosity stratification and Density stratification. Density-stratified fluid flows are defined as those that allow for density variations. When the fluid is not disturbed and when the fluid is disturbed by external forces are the two primary situations in which stratification needs to be addressed. Stratification depends almost entirely on all parameters when it is disrupted, but only on gravitational and vertical parameters when it is undisturbed. The exponential and linear distributions of height are the two main characteristics of the density in an unaltered condition. According to the literature review, the linear distribution, which takes the first two terms of expansion in exponential form, is a close approximation to the exponential distribution of density in the vertical plane.

However, because the higher-order terms are so insignificant in the exponential case, their effects on different parameters aren't very noticeable. Therefore, in this discussion, we contrast the impacts of exponential stratification with those of the linear density distribution along the direction of gravity. Interesting findings can be obtained by applying this phenomenon in a narrow channel where the width-to-length ratio is very less. In their detailed discussion of "flow of unsteady and MHD heterogeneous fluid through previous medium over a plate which is moving in slip flow condition, " Khandelwal and Jain [1] presented a perturbation solution for different flow characteristics. The issue considers the suction velocity applied perpendicular to the plate and the time-dependent, exponentially changing plate velocity. Ghione et al. assessed the parameters for predicting the flow excursion instabilities in vertical narrow rectangular channels in 2017 [2]. Song et al., 2020 investigated the rectangular narrow channel in which the plate fuel element flowed, developed a laminar flow mechanism, and ascertained the flow properties of the channel [3]. A mathematical model of oscillatory and vertically exponential density stratified viscous fluid flow that is caused by continuous suction through one side of the channel was put forth by Prasanna Venkatesh L in 2021. This study found that the linearly distributed density stratification parameter has the greatest impact on flow in the vertical plane. The outcomes of this comparison were made with those of linear stratification [4]. Prakash [5] presented "Transfer of Heat on Dusty and MHD flow of non-Newtonian heterogeneous fluid with variable viscosity in a porous medium" by expressing both the horizontal as well as vertical velocity expressions as a combination of y, t and by illustrating the variable separable substitution for deriving a class of exact solutions to fluid velocity.

The flow formulae simplify to narrow channel flows when medium Reynolds number fluids and channels with lengths considerably greater than breath are considered. The term "lubrication approximation," which has its roots primarily in the lubrication industry, refers to this simplification of momentum formulae. Other industries that are comparable to and supportive of the lubrication sector are also interested in such flows. By assuming that the flow is time-independent, Panton [6] discussed the idea of lubrication approximation issues, which we refer to as narrow channel problems for plane Poiseuille flow. Krechetnikov [7] elaborated on the various applications of narrow channel simplification to coating flows in multiple directions with clear and wet boundaries by establishing weak ellipticity that is different from the commonly accepted parabolic structures that allow lubrication analysis to capture topologies of the flow in and around points of stagnation. By taking into account pressure-dominated fluid flow, he further extended it for bearing issues and expressed the precise answer for velocity functions. Reynolds [8], from whom Beauchamp Tower's experiments led to the theory and industrial applications of the lubrication idea.

By assuming that the direction of the planes is vertical of the perpendicular axis of circulations, Krishna and Sharma [9] were able to solve the issue of "the motion of an axisymmetric body in a rotating stratified fluid confined between two parallel planes". By formulating density stratification as exponentially distributed with governing equations of motion of fluid in the polar form, Naidu [10] described the detailed solution to "stratified viscous flow between two oscillating cylinders". By taking into account the electrically conducting fluid's effects on very thin channels in both the axial and transverse directions, Prasanna Venkatesh [11] examined the "magnetohydrodynamics oscillating viscous flow in a very thin channel induced by removal or inclusion of fluid through the pervious plate. With the presumption that the density when there is no disturbance in the fluid is a function of depth only and that the flow is time-dependent, Prasanna Venkatesh presented the impact of "Stratification in Rectangular Channel Problem". When the flow channel narrows in contrast to its length, the effects of the inertial terms are very minimal for the type of flow under investigation. By contrasting the flow cases with and without the application of the lubrication approximation, Prasanna Venkatesh and Surya Prabha [12] studied the "Effect of narrow channel approximation on viscous oscillatory density stratified fluid flow through rectangular channels" taking this aspect into consideration.

In the literate population, linear distribution in an undisturbed condition, which is similar to naturally occurring exponential stratification, is the most prevalent stratification characteristic. In some research studies where exponential stratification was taken into consideration, suitable substitutions were made so that the answer could be negotiated during the evaluation process. Because of this, we generalize linearly distributed density stratified fluid flow for the same case in our current talk and take into account the flow of exponential density stratified flow of viscous oscillatory fully developed flow between parallel plates.

Mathematical Model and Solution

The flow region is located between two plates, with the right-side plate being assumed to be next to the left-side plate and separated by a distance of "h." The left-side plate is fixed to be precisely on the y-axis. The plate on the right is permeable, while the one on the other is not. Through the prior plate, the fluid is periodically suctioned out or injected in at a steady velocity. The channel's dimensions allow for the application of a narrow channel approximation because the length of the channel is significantly greater than its breadth. The initial velocity of the fluid is assumed to be constant because the fluid flow under consideration is completely developed. The fluid is divided by density in this way

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{2}$$

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \sigma_e {B_0}^2 v - \rho g \tag{3}$$

$$\rho = \rho_0(y) + \rho'(x,y,t) \tag{4}$$

$$\rho_0(y) = \rho_0' \, e^{-\beta y} \tag{5}$$

$$\frac{\partial \rho}{\partial t} = {\rho_0}' \beta e^{-\beta y} v \tag{6}$$

The following explains the notations used in the above system of equations.

(i) μ - coefficient of viscosity
(ii) $\boldsymbol{\rho}$ - density of the fluid
(iii) σ - electrical conductivity
(iv) B_0 - electromagnetic induction
(v) μ_e - magnetic permeability
(vi) $\boldsymbol{\rho}_0$'- constant density
(vii) $\boldsymbol{\rho}$'(y,t) - perturbation density
(viii) β - stratification parameter
(ix) N - Brunt – Vaisala frequency N. N^2 = βg (xi) g – gravity
(xii) v_0- initial average velocity ,$u_0 = v_0 h_0/L$.

Using mathematical simplification process after applying non-dimensionalities on (2), (3) and (6) we get

$$S_t \frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\frac{1}{Re}\frac{\partial}{\partial t}\nabla^2\left(\frac{\partial v}{\partial x}\right) + \frac{1}{{Fr_d}^2} e^{-\beta y}\frac{\partial v}{\partial x} + M^2 \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) \tag{7}$$

S_t is Strouhal number which is defined as $S_t = \frac{L}{UT} = \frac{\text{Local unsteady acceleration}}{\text{convective inertia}}$

Re is Reynold's number, Fr_d represents Froude number, the velocity of a gravity wave, $Fr_d = \frac{U}{\sqrt{gL}}$

The time component in both velocity and pressure function are chosen based of the fluid oscillation in the following manner

$u(x,y,t) = u(x,y)e^{i\omega t}$,
$v(x,y,t) = V(x,y)e^{i\omega t}$,
$and\ p(x,y,t) = p(x,y)e^{i\omega t}$

$$-S_t\omega^2\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = -\frac{i\omega}{Re}\nabla^2\left(\frac{\partial v}{\partial x}\right) + \frac{1}{Fr_d^{\,2}}e^{-\beta y}\frac{\partial v}{\partial x} + i\omega M^2\frac{\partial v}{\partial x} \quad (8)$$

We define Stream Function ψ such that

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \quad (9)$$

The values of velocity on the boundary are assumed to be

u = 0 at x=0, u = u_1 at x = h
v = 0 at x=0, v = 0 at x = h

$$\Psi = \left(u_0 - \frac{v_1 y}{h}\right) f(\eta) \quad (10)$$

Where $\eta = \frac{x}{h}$ and u_0 is the average entrance velocity. Equation (9) becomes

$$\left(D^4 - \frac{iRe}{\omega}\left(\frac{1}{Fr_d^{\,2}}e^{-\beta y_0} + i\omega M^2 - S_t\omega^2\right)D^2\right)f(\eta) = 0 \quad (12)$$

Where $D^2 = \frac{d^2}{d\eta^2}$,

$$\alpha^2 = \frac{iRe}{\omega}\left(\frac{1}{Fr_d^{\,2}}e^{-\beta y_0} + i\omega M^2 - S_t\omega^2\right)$$

$$\Rightarrow \alpha = \sqrt{\frac{iRe}{\omega}\left(\frac{1}{Fr_d^{\,2}}e^{-\beta y_0} + i\omega M^2 - S_t\omega^2\right)}$$

The equation can be reduced to a linear differential equation with a constant complex coefficient by taking specific numbers for y0 into account. There is only one option because this is a complex number. The other two roots are zero, while two of them are unique and complicated.

$$f(\eta) = c_1 + c_2\eta + c_3e^{\alpha\eta} + c_4e^{-\alpha\eta} \quad (13)$$

The transformed velocity values at the boundary in terms of *f(η)* are

$$\left.\begin{aligned} f(0) &= 0 \\ f(1) &= -1 \\ f'(0) &= 0 \\ f'(1) &= 0 \end{aligned}\right\} \quad (14)$$

The equations using (14) to determine the arbitrary constants in $f(\eta)$ are

$c_1 + c_3 + c_4 = 0$ (15)
$c_1 + c_2 + c_3 e^{\alpha} + c_4 e^{-\alpha} = -1$ (16)
$c_2 + \alpha c_3 - \alpha c_4 = 0$ (17)
$c_2 + \alpha c_3 e^{\alpha} - \alpha c_4 e^{-\alpha} = 0$ (18)

Expression for ψ and (9) after substituting the constants, the velocity components are

$$u(x,y) = -u_1\left(\frac{e^{\alpha}-1-\alpha(1+e^{\alpha})X+e^{\frac{\alpha x}{h}}-e^{\alpha}e^{-\frac{\alpha x}{h}}}{(\alpha+2)+(\alpha-2)e^{\alpha}}\right) \quad (19)$$

$$v(x,y) = -(v_0 - u_1 y)\left(\frac{-\alpha(1+e^{\alpha})+\alpha e^{\frac{\alpha x}{h}}+\alpha e^{\alpha}e^{-\frac{\alpha x}{h}}}{(\alpha+2)+(\alpha-2)e^{\alpha}}\right) \quad (20)$$

RESULTS AND DISCUSSION

The representations of the solution in the form of graphs helps in observing the various functionalities of the flow due to parameters such as time, depth, Strouhal's number, Froude number, Reynold's number and Hartmann number by analysing their influence on velocity both in transverse and axial directions. Values of the spatial variables are substituted between 0 and 1 whereas parameters like Strouhal's number is taken between 0.1 to 0.2, Froude number between 1 to 4.7, Hartmann number between 1 to 5 and Reynold's number less than 100 due to the reason that the flow is through narrow channel.

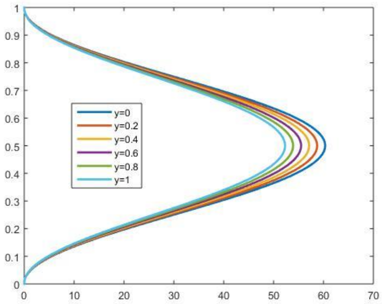

Fig. 1. Velocity in Transverse Direction for varying heights y = 0 to 1

Figure 1 to Figure 6 are drawn for depicting the solution for velocity in the transverse direction varying depth of the channel from 0 to 1, time from 0 to π, Strouhal's number between 0 to 0.2, Froude number between 1 to 1.7, Hartmann number from 1 to 5 and Reynold's number preferably less than 100 respectively. The velocity profiles are considered in y axis while the values of x in horizontal direction. We found from these representations of figures that velocity in transverse axis decreases over time.

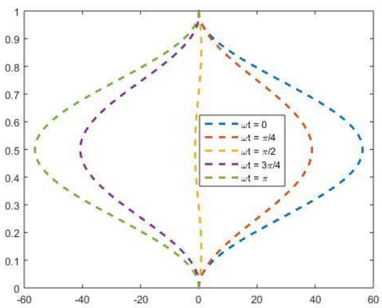

Fig. 2. Velocity in Transverse Direction for varying time at y = 1

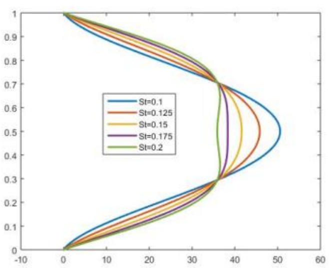

Fig. 3. Velocity in Transverse Direction for varying Strouhal's number

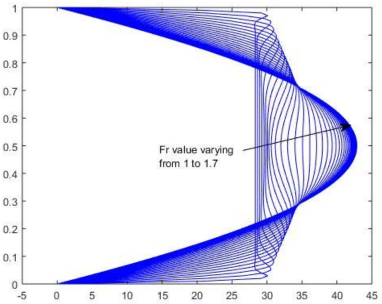

Fig. 4. Velocity in Transverse Direction for different Froude number values

The Transverse flow decreases on increase in value of y. Over the entire range of Strouhal's number the transverse velocity profiles depict a varied effect. Closer to end regions increasing while in considerably middle part of the flow region reversal of flow are observed from figure 3. Throughout the values of Froude number the darker region of velocity reveals that the flow in y direction follows similar patterns nearer to the walls and are totally the other way at mid part of the channel. Both Hartmann number as well as Reynold's number's variation also exhibits similar changes in velocity profiles in vertical direction.

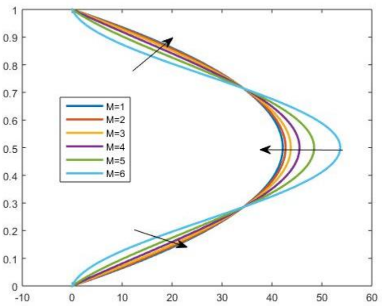

Fig. 5. Velocity in Transverse Direction for varying Hartmann number

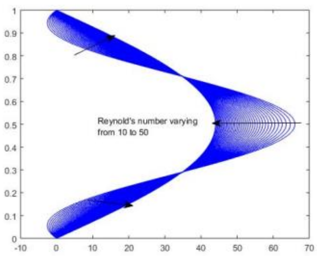

Fig. 6. Velocity in Transverse Direction for varying Reynold's number

Figure 6 shows how axial velocity behaves for change of values in time. The observation shows that as time increases, velocity increases in axial direction. While varying Reynold's number, Strouhal's number and Hartmann number for velocity profiles in horizontal direction, they all show same pattern of being symmetrical about a point at the mid-section of the channel which happens to be the midpoint of the channel. In the lower half of the values these parameters affect the velocity by decreasing them while for upper half of the values they lead to increase in values of velocity profiles but with reversal of the flow.

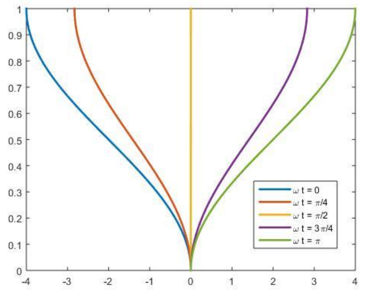

Fig. 7. Velocity in Axial Direction for varying time

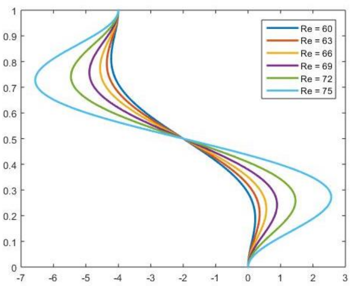

Fig. 8. Velocity in Axial Direction for varying Reynold's number

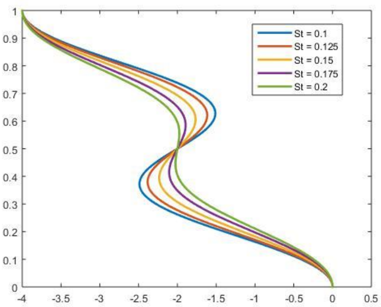

Fig. 9. Velocity in Axial Direction for varying Strouhal's number

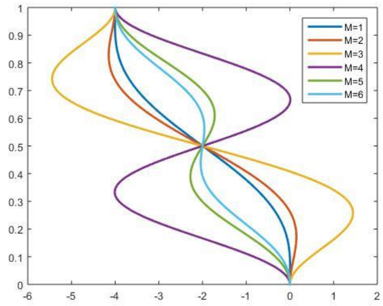

Fig. 10. Velocity in Axial Direction for varying Hartmann number

Conclusions

The combination of narrow channel and exponentially distributed stratified fluid flow with injection or suction on only one of the plates is presented in this paper. The mathematical system that represents the situation are transformed from partial differential equation to an ordinary differential equation with complex coefficient using stream function and variable separable substitutions for the velocity profiles by carefully constructed boundary conditions. The findings are inline with that of linear stratification problems that were already solved and available in literature. Certain important findings of such analysis are as follows.

- The flow increases with increases heights
- The increase in time component decreases the transverse velocity profile while increasing the axial velocity profile.
- The behaviour of transverse velocity profile closer to plates increasing the velocity while at the mid section decreasing the same is evident for Strouhal number, Froude number, Hartmann number and Reynolds number influence
- For axial velocity profile the channel gets divided into two symmetrical portions with respect the midpoint of the flow

region making the flow decrease at lower values while increasing for higher values.

- The results are in concurrent with the available ones in literature when the exponential stratification reduced to linear one by omitting the terms of higher order.

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