Modified Jackknife Estimator in Linear Regression Model

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Abstract

In multiply linear regression model, more method was proposed to improve the biased estimation. One of this method jackknife biased estimation is considered one of important methods to address the high variance and multicollinearity problems. In this paper, we propose biased estimator called modified jackknife estimator (MJE) based on the jackknife Liu type estimator (JLTE). We derive the MJE estimator as the solution of the following problem, minimize β'βsubiect to $(y - X\beta)'(y - X\beta) = c$. In section 4, we conduct a simulation study based on the mean squares error (MSE) to assess how well the new estimator performs in comparison to a few jackknife biased estimators. We demonstrate that, in comparison to several other jackknife estimators, the MJE has good qualities. Finally, using data from real-world situations, we demonstrate how well this estimator performs.

Key words: *Jackknife biased estimation, Modified jackknife estimator, Multicollinearity, Simulation study.*

1. INTRODUCTION

The following equation yields the multiple linear regression model:

$$\mathrm{A} = \mathrm{B}v + \xi, \qquad (1)$$

where v is a $(p \times 1)$ vector of the unknown parameters, p is the number of explanatory variables, A is an $(n \times 1)$ vector of the responses, B is an $(n \times p)$ matrix of the regressors variables, p is the number of the explanatory variables, and ξ is an $(n \times 1)$ vector of the random errors with $E(\xi) = 0$ and $Var(\xi) = \sigma^2 I_n$ .The jackknife method considered one of most important method to overcome this problems. Liu (2003) proposed Liu type estimator (LTE) and defined by

$$\hat{\beta}_{LTE} = (S + kI)^{-1}(B'A + d\hat{v}) \qquad k > 0, \; -\infty < d < \infty .$$

$$\hat{v}_{LTE} = F_{k.d}\hat{v} , \qquad (2)$$

where

$$F_{k.d} = ((B'A + kI)^{-1}(S - dI). \; S = (B'B)$$

We can consider the statistical properties of the any estimator according to the relationships:

$$MSE(v^*) = Var(v^*) + \big(bais(v^*)\big).\big(bias(v^*)\big)', \qquad (3)$$

where

$$Var(v^*) = E[(v^* - E(v^*)(\big(\beta - E(v^*)\big)'], \qquad (4)$$

$$Bias(v^*) = E(v^*) - v , \qquad (5)$$

where $E(v^*)$ the expected value of v^*., the statistical properties of the LTE are given by as follows:

$$bias(\hat{v}_{LTE}) = [F_{k.d} - I]v, \quad (6)$$

and variance matrix

$$Var(\hat{v}_{LTE}) = \sigma^2 F_{k.d}\, S^{-1} F_{k.d}', \quad (7)$$

$$MSE(\hat{v}_{LTE}) = \sigma^2 F_{k.d}\, S^{-1} F_{k.d}' + [F_{k.d} - I]vv'[F_{k.d} - I]' \quad (8)$$

where $S^{-1} = (B'A)^{-1}$.

More researchers introduced the jackknife biased estimators in linear regression model based on Liu type estimator. Hinkley (1977), Batah et al. (2008), Batah (2013), Türkan and Özel (2016), Özkale and Özge (2019), Asar and Kılınç. (2020), and Ugwuowo, et al. (2021). We can propose the jackknifed form of LTE. Hinkley (1977) stated that with few exceptions, the jackknife had been applied to balanced models. They used the weighted jackknife procedure. The jackknife Liu type estimator (JLTE) is given by as follows

$$\hat{v}_{JLTE}(k,d) = (2I - F_{k.d})F_{k.d}\, \hat{v}. \quad (9)$$

The properties of JLTE is given by as follows:

$$bias\left(\hat{v}_{JLTE}(k,d)\right) = (I - F_{k.d})^2 v \quad (10)$$

$$Var\left(\hat{v}_{JLTE}(k,d)\right) = \sigma^2\big((2I - F_{k.d})\big)F_{k.d}\, S^{-1}\, F_{k.d}{}'\big((2I - F_{k.d})\big)'. \quad (11)$$

The MSE of the JLTE is given by as follows:

$$MSE\left(\hat{v}_{JLTE}(k,d)\right) = \sigma^2 F_{k.d}\, S^{-1}\, F_{k.d}{}' + (F_{k.d} - I)vv'(F_{k.d} - I). \quad (12)$$

Yıldız (2018) proposed a new estimator for β called modified jackknifed Liu-type estimator (MJLTE) is given by as follows :

$$\hat{\beta}_{MJLTE}(k,d) = [I - (k+d)^2(S + kI)^{-2}][I - (k+d)(S + kI)^{-1}]. \quad (13)$$

The statistical properties of the MJLTE is given by as follows.

$$bais\left(\hat{v}_{MJLTE}(k,d)\right) = -(k+d)(s + kI)^{-1}M(S + kI)^{-1}v. \quad (14)$$

$$Var\left(\hat{v}_{MJLTE}(k,d)\right) = \sigma^2 \varphi S^{-1} \varphi'. \quad (15)$$

$$MSE\left(\hat{v}_{MJLTE}(k,d)\right) = \sigma^2 \varphi S^{-1}\varphi' + (k + d)(S + kI)^{-1}M(S + kI)^{-1}v'v\,[(S + kI)^{-1}M(S + kI)^{-1}]', \quad (16)$$

where $\varphi = (2I - F_{k.d})(F_{k.d})^2$ and $M = I + F_{k.d} - (F_{k.d})^2$.

The goal of MJE estimator is to improve the results and address problem in other estimators. In section2, We show that, the statistical properties of MJE estimator . In Section 3, some theorms show that , the performance of the MJE estimator . We studied the simulation of the MJE with some jackknife biased estimators such as (OLS) , (JLTE) and (MJLTE) in Section 4 . Section 5 contain illustrated example of the results with diagrams.

2. New Estimator.

we propose new jackknife biased estimator called modified jackknife estimator based on the jackknife Liu type estimator. We obtained the proposed estimator by driving and as the answer to the minimization problem for the corresponding function:

$$\Phi = \left(\beta - \hat{\beta}_{JLTE}(k,d)\right)'\left(\beta - \hat{\beta}_{JLTE}(k,d)\right) + \frac{1}{k}(y - X\beta)'(y - X\beta) - c]. \quad (17)$$

From (17) we have

$$\Phi = \left(v'v - 2v\hat{v}_{JLTE}(k,d) + 2\hat{v}_{JLTE}(k,d)\right) + \frac{1}{k}\big((A'A - 2v'B'A + v'B'Bv') - c\big)$$

$$\Phi = \left(v'v - 2v\hat{v}_{JLTE}(k,d) + 2\hat{v}_{JLTE}(k,d)\right) + \left(\left(\frac{1}{k}A'A - \frac{2}{k}v'B'A + \frac{1}{k}v'B'Bv\right) - \frac{c}{k}\right),$$

where $\frac{1}{k}$ is a Lagrangian multiplier, We differentiate the Φ with respect v , we obtain the following equations:

$$\frac{\partial \Phi}{\partial \beta} = 2v - 2\hat{v}_{JLTE}(k,d) + \left[-\frac{2}{k}(B'A) + \frac{2}{k}B'Bv\right]$$

$$\Rightarrow 2v - 2\hat{v}_{JLTE}(k,d) + \frac{1}{k}[-2(B'A) + 2B'Bv] = 0\,. \qquad (18)$$

From (18) we obtain:

$$\frac{(B'Bv + kv) - \left(\left(B'A + k\hat{v}_{JLTE}(k,d)\right)\right)}{k} = 0$$

$$\Rightarrow (B'Bv + kv) - \left(B'A + k\hat{v}_{JLTE}(k,d)\right) = 0$$

We obtain the modified jackknife estimator as follows :

$$\hat{v}_{MJE}(k,d) = (B'B + kI)^{-1}(B'A + k\hat{v}_{JLTE}(k,d).$$

$$\hat{v}_{MJE}(k,d) = S_k^{-1}\left(B'A + k\hat{v}_{JLTE}(k,d)\right). \qquad (19)$$

Now, we can to show that, the properties of MJE estimator. the statistical properties of the MJE estimator is given by as follows:

$$bias\left(\hat{v}_{MJE}(k,d)\right) = E\left(\hat{v}_{MJE}(k,d)\right) - v$$
$$= E\left(S_k^{-1}\left(B'A + k\hat{v}_{MJE}(k,d)\right)\right) - v$$
$$= S_k^{-1}(B'B + k((2I - F_{k.d})F_{k.d})v - v$$
$$= \left(M_{k,d} - I\right)\beta, \qquad (20)$$

where $M_{k,d} = S_k^{-1}(B'B + k((2I - F_{k.d})F_{k.d})$.

$$Var\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2 M_{k,d} S_k^{-1} M_{k,d}. \qquad (21)$$

The Mean Square Error (MSE) is calculated as follows:

$$MSE\left(\hat{v}_{MJE}(k,d)\right) = Var\left(\hat{v}_{MJE}(k,d)\right) + bias\left(\hat{v}_{MJE}(k,d)\right) bias\left(\hat{v}_{MJE}(k,d)\right)'$$

$$= \sigma^2 M_{k,d} S_k^{-1} M_{k,d} + \left(M_{k,d} - I\right) vv' \left(M_{k,d} - I\right). \qquad (22)$$

The Scalar Mean Square Error of the MJE is given by as follows:

$$SMSE\left(\hat{v}_{MJE}(k,d)\right) = tr\left(M_{k,d} S_k^{-1} M_{k,d}\right) + tr(\left(M_{k,d} - I\right) v'v \left(M_{k,d} - I\right). \qquad (23)$$

3. The MJE's Performance in Relation to Other Estimators.

In this section, we compare the MJE estimator with LTE, JLTE and MJLTE estimator in terms of MSE as follows:

3.1 Comparison between the MJE and LTE estimators

By MSE, performance MJE estimator is better than LTE. The following gives the difference between the MJE estimator and the LTE estimator:

$$MSE(\hat{v}_{LTE}) - MSE\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2\left[F_{k.d}\, S^{-1} F'_{k.d} - M_{k,d} S_k^{-1} M_{k,d}\right] + C_1 C_1' - C_2 {C_2}'$$

where $C_1 C_1'$ the bias of LTE and $C_2 C_2$ the bias of MJE.

Theorem 1: If $k > 0$ and $d > 0$ the MJE is better than of the LTE by using the MSE criterion, that is $MSE(\hat{v}_{LTE}) - MSE\left(\hat{v}_{MJE}(k,d)\right) > 0$ if and only if $C_2' \sigma^2\, F_{k.d}\, S^{-1} F'_{k.d} - M_{k,d} S_k^{-1} M_{k,d} C_2 < 1$.

Proof: The difference variance between the variance of MJE and the LTE estimators is given by :

$$Var(\hat{v}_{LTE}) - Var\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2\left[F_{k.d}\, S^{-1} F'_{k.d} - M_{k,d} S_k^{-1} M_{k,d}\right]$$

$$:Var(\hat{v}_{LTE}) - Var\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2 diag\left\{\frac{(\lambda_i-d)^2}{(\lambda_i+k)^2\lambda_i} - \frac{\lambda_i(\lambda_i+k)^2+(\lambda_i+k)(\lambda_i-d)-(\lambda_i-d)^2}{(\lambda_i+k)^3}\right\}_{i=1}^{p}$$

Therefore $F_{k.d}\, S^{-1}F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}$ is positive definite if and only if

$$(\lambda_i-d)^2(\lambda_i+k)^4 - [(\lambda_i+k)^2\lambda_i\lambda_i(\lambda_i+k)^2 + (\lambda_i+k)(\lambda_i-d) - (\lambda_i-d)^2] > 0$$

Consequently, $\left[F_{k.d}\, S^{-1}F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}\right]$ is (p.d) . The proof is completed.

3.2Comparison between the MJE and JLTE estimators

The difference MSEM between the variance of MJE and the JLTE estimators is given by :

$$MSE(\hat{v}_{JLTE}) - MSE\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2\left[F_{k.d}\, S^{-1}\, F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}\right] + C_3C'_3 - C_2{C_2}'$$

where the bias$C_3C'_3$ of JLTE.

Theorem 2: If $k > 0$ and $-\infty < d < \infty$, the performance of the MJE is the best estimator by using the MSE criterion, that is $MSE(\hat{v}_{JLTE}) - MSE\left(\hat{v}_{MJE}(k,d)\right) > 0$ if and only if $C'_2F_{k.d}\, S^{-1}\, F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}C_2 < 1$

Proof : The difference variance between the variance of MJE and the LTE estimators is given by :

$$Var(\hat{v}_{JLTE}) - Var\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2\left[F_{k.d}\, S^{-1}\, F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}\right]$$

$$Var(\hat{v}_{JLTE}) - Var\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2 diag\left\{\frac{(\lambda_i-d)^2}{(\lambda_i+k)^3\lambda_i} - \frac{\lambda_i(\lambda_i+k)^2+(\lambda_i+k)(\lambda_i-d)-(\lambda_i-d)^2}{(\lambda_i+k)^3}\right\}_{i=1}^{p}$$

Therefore $\sigma^2\left[F_{k.d}\, S^{-1}\, F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}\right]$is positive definite if and only if

$$(\lambda_i-d)^2(\lambda_i+k)^3 - (\lambda_i+k)^3\lambda_i\lambda_i(\lambda_i+k)^2 + (\lambda_i+k)(\lambda_i-d) - (\lambda_i-d)^2 > 0$$

Consequently, $F_{k.d}\, S^{-1}\, F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}$is (p.d) . The proof is completed.

3.3 Comparison between the MJE and MJLTE estimators

The difference MSEM between the variance of MJE and the MJLTE estimators is given by:

$$MSE(\hat{v}_{MJLTE}) - MSE\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2\left[\varphi S^{-1}\varphi' - M_{k,d}S_k^{-1}M_{k,d}\right] + C_4C'_4 - C_2C'_2$$

where $C_4C'_4$ the bias of MJLTE .

Theorem 3: If $k > 0$ and $-\infty < d < \infty$ the performance of the MJE is better than of the MJLTE estimator using the MSE criterion, that is $MSE(\hat{v}_{MJLTE}) - MSE\left(\hat{v}_{MJE}(k,d)\right) > 0$ if and only if $C'_2\sigma^2\left[\varphi S^{-1}\varphi' - M_{k,d}S_k^{-1}M_{k,d}\right]C_2 < 1$

Proof : The difference variance between the variance of MJE and the MJLTE estimators is given by :

$$Var(\hat{v}_{MJLTE}) - Var\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2\left[\varphi S^{-1}\varphi' - M_{k,d}S_k^{-1}M_{k,d}\right]$$

$$Var(\hat{v}_{MJLTE}) - Var\left(\hat{v}_{MJE}(k,d)\right) = \sigma^2 diag\left\{\frac{(\lambda_i + k + 2d_i)^2(k+d)(2\lambda_i + k + d_i)}{(\lambda_i + (\lambda_i + k)^4)} - \frac{\lambda_i(\lambda_i + k)^2 + (\lambda_i + k)(\lambda_i - d) - (\lambda_i - d)^2}{(\lambda_i + k)^3}\right\}_{i=1}^{p}$$

Therefore $\sigma^2\varphi S^{-1}\varphi' - M_{k,d}S_k^{-1}M_{k,d}$ is positive definite if and only if

$$(\lambda_i + k + 2d_i)^2(k+d)(2\lambda_i + k + d_i)(\lambda_i + k)^3 - (\lambda_i + (\lambda_i + k)^4)\lambda_i\lambda_i(\lambda_i + k)^2 + (\lambda_i + k)(\lambda_i - d) - (\lambda_i - d)^2 > 0$$

Consequently, $\sigma^2\left[\varphi S^{-1}\varphi' - M_{k,d}S_k^{-1}M_{k,d}\right]$ is (p.d).

4. Simulation Study

Using the Matlab software, we wish to compare the performance of the proposed estimator to several other jackknife biased estimators in order to demonstrate how well it performs. This study compares the MJE estimator's performance against a few other constrained estimators that are already in use. This simulation is designed to evaluate the performance of the estimators OLS, JLTE, MJLTE, and MJE when the regressors are highly correlated. The following equation was used to generate the matrix B, according to (Najarian, et al. ,2013):

$$B_{ij} = (1-\mu^2)^{1/2}Z_{ij} + \mu Z_{ip}, \quad i = 1,2,\dots n \;\; j = 1,2,\dots p \qquad (24)$$

where μ stands for any two variables' correlation with one another and Z_{ij} independent standard normal pseudo-random numbers. The standardized nature of these variables allows for the correlation form of $B'B$. The response variable A is considered by:

$$A_i = v_0 + v_1 b_{i1} + v_2 b_{i2} \dots\dots + v_p b_{ip} + e_i, \quad i = 1,2,\dots\dots,n \qquad (25)$$

Where e_i independent, identically distributed (i.i.d.) random variables make up N(0, σ^2) Therefore, it will be assumed that (25) has zero intercept. Additionally, p=5 while σ are chosen as the explanatory values (1, 5, 10). The correlation coefficient μ will be set at (0.85, 0.95, and 0.99) with the sample size is (n = 50, 100, 150). According to the condition $\nu'\nu = 1$ that the matrix's largest eigenvalue must be greater than one, the coefficients ν_1, ν_2, …, ν_p are chosen as the eigenvectors corresponding to that value. Thus, sets of B`s are created for all n, σ, p, ν and μ. By creating new error terms, the experiment was repeated 5000 times. Here is how to calculate estimated mean square error (EMSE):

$$EMSE(\nu^{**}) = \frac{1}{5000}\sum_{i=1}^{5000}(\nu^{**} - \nu)'(\nu^{**} - \nu),$$

Hence, ν^{**} any estimators would be (OLS, JLTE, MJLTE or MJE).

Table 1: Calculated MSE under $n = 50$, $\mu = 0.85$, $p = 5$

σ	k	OLS	JLTE	MJLTE	MJE
1	k_{HK}	0.317104	0.810276	0.780279	0.310975
	k_{HKB}	0.317104	0.797312	0.737419	0.315044
	k_{LW}	0.317104	0.81046	0.78132	0.310937
	k_{HSL}	0.317104	0.883744	0.983603	0.776964
	k_{HMO}	0.317104	0.677734	0.687807	0.361712
	k_{AM}	0.317104	0.677562	0.688018	0.361831
	k_{GM}	0.317104	0.811334	0.786805	0.310781
	k_{HK}	0.982812	0.73197	0.623676	0.812263

5	k_{HKB}	0.982812	0.646336	0.505237	0.647123
	k_{LW}	0.982812	0.627783	0.492059	0.634074
	k_{HSL}	0.982812	0.96031	0.998906	0.948775
	k_{HMO}	0.982812	0.525543	0.741905	0.555171
	k_{AM}	0.982812	0.751278	0.947603	0.716244
	k_{GM}	0.982812	0.742254	0.652431	0.866829
10	k_{HK}	1.071423	0.643824	0.57452	1.239995
	k_{HKB}	1.071423	0.576455	0.527847	1.127572
	k_{LW}	1.071423	0.649136	0.583433	1.248783
	k_{HSL}	1.071423	0.998256	0.999998	0.998254
	k_{HMO}	1.071423	0.634303	0.841025	0.775105
	k_{AM}	1.071423	0.518562	0.578064	0.944926
	k_{GM}	1.071423	0.660731	0.609722	1.270458

Table 2: Calculated MSE under $n = 50$, $\mu = 0.85$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	0.343045	0.807172	0.713273	0.334269
	k_{HKB}	0.343045	0.742367	0.658552	0.340111
	k_{LW}	0.343045	0.807437	0.713635	0.33426
	k_{HSL}	0.343045	0.646439	0.657095	0.329145
	k_{HMO}	0.343045	0.678245	0.644597	0.335158
	k_{AM}	0.343045	0.66708	0.646013	0.332641
	k_{GM}	0.343045	0.813943	0.723144	0.334235
5	k_{HK}	0.754714	0.816453	0.72587	0.671644
	k_{HKB}	0.754714	0.748039	0.710356	0.66237
	k_{LW}	0.754714	0.731723	0.713559	0.662793
	k_{HSL}	0.754714	0.724014	0.71692	0.66318
	k_{HMO}	0.754714	0.732468	0.713314	0.662762
	k_{AM}	0.754714	0.719015	0.720219	0.663514
	k_{GM}	0.754714	0.87579	0.77559	0.718701
10	k_{HK}	1.236318	0.652418	0.489492	0.906426
	k_{HKB}	1.236318	0.537064	0.451619	0.808685
	k_{LW}	1.236318	0.466349	0.462679	0.747501
	k_{HSL}	1.236318	0.44556	0.496471	0.708875
	k_{HMO}	1.236318	0.470218	0.460455	0.751839
	k_{AM}	1.236318	0.446335	0.492199	0.712391
	k_{GM}	1.236318	0.715262	0.537817	0.982808

Table 3: Calculated MSE under $n = 50$, $\mu = 0.99$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	0.31827	2.464875	1.950838	1.132172
	k_{HKB}	0.31827	2.276126	1.884201	0.697428
	k_{LW}	0.31827	2.131606	1.852025	0.491558
	k_{HSL}	0.31827	2.113848	1.848976	0.472586
	k_{HMO}	0.31827	2.174838	1.860212	0.543241
	k_{AM}	0.31827	2.084591	1.844315	0.444078
	k_{GM}	0.31827	2.470278	1.953235	1.147917
5	k_{HK}	1.208908	1.117427	1.111569	0.59839
	k_{HKB}	1.208908	1.114584	1.110428	0.57682
	k_{LW}	1.208908	1.112915	1.106097	0.581456
	k_{HSL}	1.208908	1.0691	1.026096	0.852214
	k_{HMO}	1.208908	1.11245	1.103258	0.590862
	k_{AM}	1.208908	1.113448	1.108528	0.575059

	k_{GM}	1.208908	1.120754	1.112046	0.628453
10	k_{HK}	1.122356	2.835006	2.790421	1.202629
	k_{HKB}	1.122356	2.818098	2.735703	1.180704
	k_{LW}	1.122356	2.56539	1.957153	1.544611
	k_{HSL}	1.122356	2.611162	2.040751	1.511529
	k_{HMO}	1.122356	2.787239	2.531787	1.281223
	k_{AM}	1.122356	2.773835	2.472377	1.311464
	k_{GM}	1.122356	2.843823	2.798117	1.233834

Table 4: Calculated MSE under $n = 100$, $\mu = 0.85$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	0.265033	0.662221	0.645477	0.248516
	k_{HKB}	0.265033	0.65887	0.627149	0.251397
	k_{LW}	0.265033	0.662228	0.645534	0.248509
	k_{HSL}	0.265033	0.600576	0.785474	0.401142
	k_{HMO}	0.265033	0.575156	0.550737	0.292662
	k_{AM}	0.265033	0.55005	0.569058	0.301727
	k_{GM}	0.265033	0.662362	0.646677	0.24837
5	k_{HK}	0.77472	1.361409	1.296297	0.94268
	k_{HKB}	0.77472	1.324232	1.219319	0.939848
	k_{LW}	0.77472	1.339491	1.243662	0.941944
	k_{HSL}	0.77472	1.258986	1.156086	0.919943
	k_{HMO}	0.77472	1.285679	1.176757	0.929837
	k_{AM}	0.77472	1.252032	1.151407	0.917122
	k_{GM}	0.77472	1.369774	1.333343	0.942293
10	k_{HK}	1.146842	1.125806	1.074725	0.878287
	k_{HKB}	1.146842	1.093454	1.041954	0.867982
	k_{LW}	1.146842	1.110409	1.055692	0.869392
	k_{HSL}	1.146842	1.032756	1.015017	0.913187
	k_{HMO}	1.146842	1.059489	1.02578	0.88024
	k_{AM}	1.146842	1.056708	1.02479	0.882194
	k_{GM}	1.146842	1.144594	1.122891	0.923641

Table 5: Calculated MSE under $n = 100$, $\mu = 0.95$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	0.731287	0.722659	0.679662	0.306241
	k_{HKB}	0.731287	0.697785	0.634382	0.314621
	k_{LW}	0.731287	0.72268	0.679726	0.306236
	k_{HSL}	0.731287	0.609612	0.715338	0.380152
	k_{HMO}	0.731287	0.601921	0.613471	0.334392
	k_{AM}	0.731287	0.593713	0.635226	0.340796
	k_{GM}	0.731287	0.724467	0.68536	0.305833
5	k_{HK}	1.209878	1.165454	1.141849	0.761083
	k_{HKB}	1.209878	1.158944	1.129965	0.781093
	k_{LW}	1.209878	1.166803	1.143662	0.758309
	k_{HSL}	1.209878	1.145908	1.100758	0.832846
	k_{HMO}	1.209878	1.15477	1.120357	0.798125
	k_{AM}	1.209878	1.158915	1.129901	0.781205
	k_{GM}	1.209878	1.201584	1.178938	0.738977
	k_{HK}	2.70207	2.657033	2.502073	1.22869
	k_{HKB}	2.70207	2.573689	2.37786	1.09055

10	k_{LW}	2.70207	2.42279	2.250625	0.947461
	k_{HSL}	2.70207	2.417979	2.246377	0.944082
	k_{HMO}	2.70207	2.485554	2.300068	0.999265
	k_{AM}	2.70207	2.459589	2.280266	0.976441
	k_{GM}	2.70207	2.670281	2.533686	1.26608

Table 6: Calculated MSE under $n = 100$, $\mu = 0.99$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	0.626298	0.526944	0.494047	0.31856
	k_{HKB}	0.626298	0.502093	0.491792	0.311229
	k_{LW}	0.626298	0.526964	0.494052	0.318561
	k_{HSL}	0.626298	0.489573	0.522021	0.307537
	k_{HMO}	0.626298	0.489304	0.50164	0.302645
	k_{AM}	0.626298	0.488747	0.509572	0.303904
	k_{GM}	0.626298	0.530429	0.494924	0.318663
5	k_{HK}	2.342715	1.548775	1.21931	1.312315
	k_{HKB}	2.342715	1.337437	1.177472	0.936316
	k_{LW}	2.342715	1.197641	1.165116	0.645015
	k_{HSL}	2.342715	1.205678	1.165972	0.662228
	k_{HMO}	2.342715	1.237895	1.168454	0.732118
	k_{AM}	2.342715	1.202201	1.165628	0.654735
	k_{GM}	2.342715	1.57825	1.228201	1.359738
10	k_{HK}	5.310773	4.60581	3.999514	1.91025
	k_{HKB}	5.310773	4.373545	3.891496	1.277898
	k_{LW}	5.310773	3.87962	3.700874	0.792957
	k_{HSL}	5.310773	3.897997	3.724558	0.787347
	k_{HMO}	5.310773	4.116674	3.814819	0.898923
	k_{AM}	5.310773	4.137641	3.820259	0.919821
	k_{GM}	5.310773	4.633754	4.016178	2.011022

Table 7: Calculated MSE under $n = 150$, $\mu = 0.85$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	0.869685	0.867971	0.842453	0.294614
	k_{HKB}	0.869685	0.863935	0.819397	0.297705
	k_{LW}	0.869685	0.867993	0.842635	0.294593
	k_{HSL}	0.869685	0.715758	0.838998	0.447811
	k_{HMO}	0.869685	0.788908	0.704663	0.329132
	k_{AM}	0.869685	0.779424	0.699654	0.331638
	k_{GM}	0.869685	0.868229	0.844616	0.294372
5	k_{HK}	1.124593	1.106089	1.077464	0.850036
	k_{HKB}	1.124593	1.084368	1.060723	0.862531
	k_{LW}	1.124593	1.087303	1.062662	0.859453
	k_{HSL}	1.124593	1.064166	1.041011	0.90503
	k_{HMO}	1.124593	1.073084	1.051757	0.880918
	k_{AM}	1.124593	1.072368	1.051028	0.882551
	k_{GM}	1.124593	1.123151	1.110827	0.871896
10	k_{HK}	1.427243	1.419208	1.382672	1.029956
	k_{HKB}	1.427243	1.397865	1.338219	1.02159
	k_{LW}	1.427243	1.420649	1.387053	1.033014
	k_{HSL}	1.427243	1.09264	1.013279	1.075843
	k_{HMO}	1.427243	1.317725	1.196408	1.094951
	k_{AM}	1.427243	1.286799	1.151955	1.112838
	k_{GM}	1.427243	1.426352	1.413075	1.060651

Table 8: Calculated MSE under $n = 150$, $\mu = 0.95$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	0.750307	0.743083	0.700389	0.302082
	k_{HKB}	0.750307	0.728364	0.664723	0.310128
	k_{LW}	0.750307	0.743097	0.700436	0.302073
	k_{HSL}	0.750307	0.719627	0.883696	0.524621
	k_{HMO}	0.750307	0.614008	0.60819	0.326274
	k_{AM}	0.750307	0.626997	0.603846	0.327347
	k_{GM}	0.750307	0.743968	0.703554	0.301492
5	k_{HK}	1.838225	1.805122	1.723878	0.786587
	k_{HKB}	1.838225	1.753979	1.666588	0.739379
	k_{LW}	1.838225	1.742731	1.657591	0.731584
	k_{HSL}	1.838225	1.669055	1.574328	0.730913
	k_{HMO}	1.838225	1.704381	1.626817	0.713481
	k_{AM}	1.838225	1.696579	1.618894	0.712709
	k_{GM}	1.838225	1.821564	1.756092	0.810062
10	k_{HK}	1.470133	1.402241	1.337773	0.810504
	k_{HKB}	1.470133	1.372207	1.31977	0.804535
	k_{LW}	1.470133	1.353099	1.307682	0.808656
	k_{HSL}	1.470133	1.357695	1.311017	0.806787
	k_{HMO}	1.470133	1.364603	1.315396	0.80515
	k_{AM}	1.470133	1.362897	1.314364	0.805445
	k_{GM}	1.470133	1.450629	1.392029	0.859684

Table 9: Calculated MSE under $n = 150$, $\mu = 0.99$, $p = 5$

σ	k	OLS	$JLTE$	$MJLTE$	MJE
1	k_{HK}	1.263531	1.244183	1.144275	0.394581
	k_{HKB}	1.263531	1.227096	1.100785	0.36838
	k_{LW}	1.263531	1.244457	1.145122	0.395167
	k_{HSL}	1.263531	1.101864	0.959973	0.336202
	k_{HMO}	1.263531	1.193624	1.043583	0.345334
	k_{AM}	1.263531	1.200572	1.053556	0.348414
	k_{GM}	1.263531	1.246107	1.150366	0.398861
5	k_{HK}	1.257794	1.141827	1.129315	0.675387
	k_{HKB}	1.257794	1.135823	1.126325	0.678893
	k_{LW}	1.257794	1.136668	1.127037	0.677593
	k_{HSL}	1.257794	1.08563	1.034749	0.896731
	k_{HMO}	1.257794	1.131466	1.115933	0.70422
	k_{AM}	1.257794	1.136098	1.126578	0.678411
	k_{GM}	1.257794	1.177028	1.137736	0.688484
10	k_{HK}	2.348196	2.110242	2.011122	0.87661
	k_{HKB}	2.348196	2.037045	1.990542	0.76884
	k_{LW}	2.348196	2.00571	1.941171	0.769016
	k_{HSL}	2.348196	1.740596	1.357551	1.15294
	k_{HMO}	2.348196	1.98711	1.840952	0.841526
	k_{AM}	2.348196	1.892076	1.582226	1.021695
	k_{GM}	2.348196	2.13228	2.018538	0.925129

4.1 The discussion of simulation results

Through the simulation study , we show that, the performance of the MJE compared with some jackknife biased estimators. From Table 1 to 9, we show that the performance of this estimator for all cases of n , $\mu$and σ.

1.From Table 1 when ($n = 50$, $\mu = .85$, $\sigma = 1$) the MJE estimator has minimum mean square error EMSE. While $\sigma = 5, 1015$ the performance of the MJLTE is the best comparing of other estimators.

2.From Table 2 when ($n = 50,\ \mu = 95, .\sigma = 1, 5$), the MJE estimator is superior to of any estimator biased estimators .

3. From Table 3 when ($n = 50\ , \mu = .99\ , \sigma = 1$) the performance the OLS estimator is the best. While $\sigma = 5, , 10$ the MJE estimator has minimum EMSE comparing with other estimator. So that the MJE is superior to of any other estimator.

4.From Table 4, 5, 6, 7, 8 and 9) for all cases of sample size , standard deviation σ and correlation coefficient ϒ , performance of the MJE estimator is best because the MJE has minimum EMSE comparing of jackknife biased estimators .

Through the simulation study in this section, it becomes evident that, when the sample size increases the performance of jackknife biased estimators becomes the best.

5. Numerical Example

The data are taken from Akdeniz (2003) and , Gruber (2017). This data expresses of the gross national product that applied wildly. The goal of utilizing this data is to demonstrate how well the MJE estimator performs in comparison to some jackknife estimators and establish that the estimator has good features. The OLS, MJLTE, JLTE, and MJE estimators are compared using the Scalar Mean Square Error (SMSE) standards.

By using real life data from this example, we can prove that theorm1, theorm2 and theorm3 as following

$\left[F_{k.d}\, S^{-1}F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}\right] = 8.2637$ that means

$\left[F_{k.d}\, S^{-1}F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}\right]$ is positive definite.

$[F_{k.d}\ S^{-1}\ F'_{k.d} - M_{k,d}S_k^{-1}M_{k,d}] = 148.4539$ is positive definite.

$\left[\varphi S^{-1}\varphi' - M_{k,d}S_k^{-1}M_{k,d}\right] = 21.9367$ is positive definite.

Table 10: SMSE of the OLS, MJLTE, JLTE and the MJE estimators of the for different values *k*

k	OLS	$JLTE$	$MJLTE$	MJE
0.161	602.4197	52.9937	46.8706	36.5654
0.243	602.4197	34.0476	22.0822	23.5986
0.050	602.4197	98.4053	460.2847	96.5987
0.10	602.4197	67.0358	116.3531	54.6033
0.20	602.4197	41.2467	31.1122	28.8552
0.25	602.4197	168.6893	21.3857	20.2116

Table 10, show that, the MJE estimator has minimum estimated SMSE for all different values k and the performance of the MJE estimator is better than of any jackknife biased estimator. and this is clear if we can see the Figure 1, 2 ,3 and 4.

Figure 1: SMSE of OLS, MJLTE, JLTE and the MJE estimators for different k.

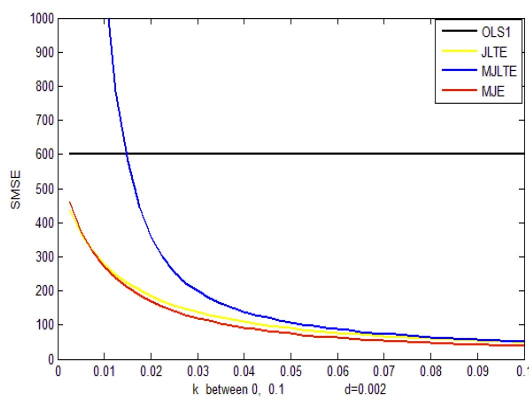


Figure 2: SMSE of OLS, MJLTE, JLTE and the MJE estimators for different k.

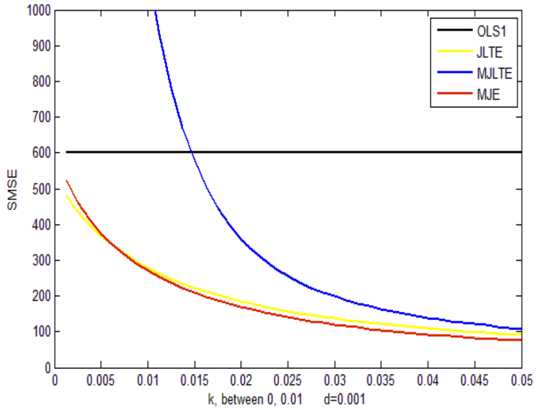


Figure 3: SMSE of OLS, MJLTE, JLTE and the MJE estimators for different k.

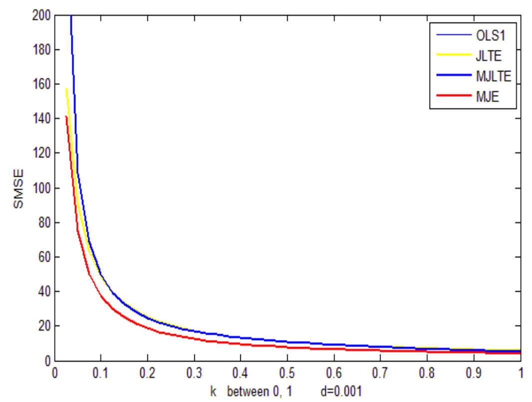
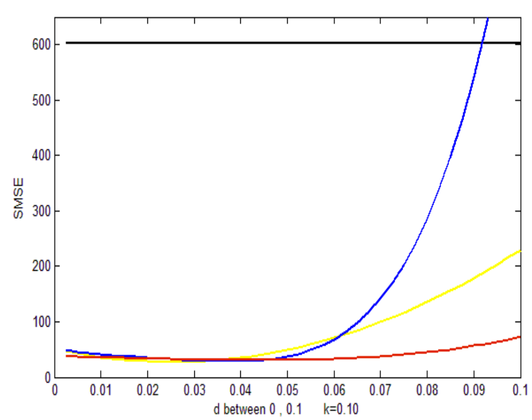


Figure 4: SMSE of OLS, MJLTE, JLTE and the MJE estimators for different d and k=0.10.



6. Conclusion

The simulation study in section 4 demonstrates that the MJE estimator has the lowest EMSE when compared to other jackknife biased estimators, indicating that it is superior to all jackknife biased estimators overall. It is also evident from some figures that the MJE estimator has good properties when compared to some jackknife estimators.

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