Mathematical Modeling for Fluid Flow In Open Rectangular Channel

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Abstract
The discharge-depth relationships in regular channels have been rarely investigated to make it more useful flow measuring devices. The authors, for the first time, have proposed a theoretical investigation on the flow in open rectangular channel and it is found that even in rectangular channel, it is possible to get a near linear depth-discharge relationship. In this paper, the flow parameters are determined through a new general optimization procedure presented. It is found that the near linear depth-discharge relationship is valid from Y_A(b) to Y_B(b), within a deviation of ±2 percent error, where b is the half base width of the channel. The proposed linear equation is given by $q_L = m\,(y - Q_c)C\sqrt{S}b^{3/2}$, where q_L is the discharge in the channel, y is the flow depth, C is Chezzy's Constant and S is the channel bed slope. Chezzy's C can also be substituted by Manning's n by the simple equation $C = \frac{1}{n}R^{1/6}$, where R the Hydraulic mean radius could be computed as $R = \frac{a}{P}$, a being the cross-sectional area of flow and P the wetted perimeter.
The significance of proposed research is that, the shift is from measurement of discharge through computations to direct reading of the discharge with a piezometer.

1. INTRODUCTION

Flow measurement is a field in engineering which, of late, has assumed extreme importance, due to the improved awareness about water management. In most of the cases, for a predetermined discharge, the channel dimensions are designed.

Least investigative work is done on the reverse process of determining discharge from the known channel dimensions. The discharge measurement is a much unpredictable task which depends on multiple factors. The major factor hindering the measurement is being the induced loss of head. Generally, the measuring flumes are designed by introducing a head loss between two sections. The cross-section of a channel may be closed or open at the top. The channels that have an open top are referred to as open channels while those with a closed top are referred to as closed conduits [9].

The term natural channel denotes to all channels which have been developed by natural processes (also referred to as drainages) and have not been significantly improved by humans. Free surface at atmospheric pressure is an important characteristic of flow in open channel [1, 3, 4, 12].

Discharge Measurement in open channels is the main concerns in irrigation, environmental and hydraulic engineering field. Flow measuring structures, which are typically used to operate as a control in the channel, provide a special link between the flow discharge and up-stream head [1, 6, 10, 11].

In this paper, a theoretical investigation on the flow in open rectangular channel has been done; it is found that even in rectangular channel it is possible to get a near linear depth-discharge relationship. The major thrust in the paper is that, the shift is from measurement to direct reading of the discharge. Something similar to reading the head value we can read the discharge value (similar to rotameters in closed conduits).

2. LITERATURE SURVEY

For the purpose of Optimization technique, it is attempted to look into certain available research work on the above technique. It was Allen P. Cowgill (1944) [2] who attempted on this technique to build a relationship between the weir profile and the head-discharge relationship. Later Keshava Murthy and Shesha Prakash (1995) [7,8] improved and found new faster and better methods to get the linear, quadratic, logarithmic, exponential and any given power, thereby mastering the technique of optimization. They developed a new numerical and algebraic optimization procedures to obtain the flow and weir parameters.

3. FORMULATION OF THE PROBLEM

Cowgill and Banks (Cowgill, 1944) [2] proposed the relationship between the weir profile and the discharge-head relationship as given in Figs. 1 and 2 where the equations used are as below:

$y = f(x)$ (Weir geometry)

$Q = B * h^m$ (Flow relationship)

Q being the discharge, B a proportionality constant and h head above the weir crest.

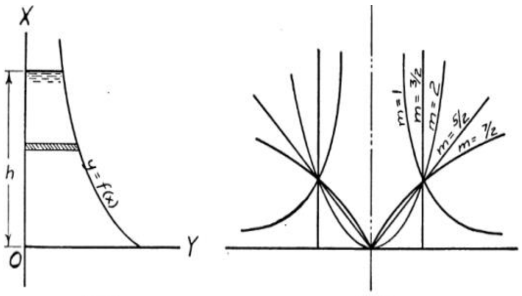

FIG. 1. FIG. 2.

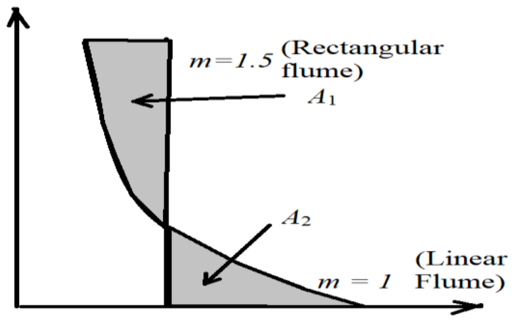

Fig 3: Definition Sketch

In the proposed research, the weir is considered to be kept at zero crest height and extending to the entire geometry of the channel. With this assumption, the theory of sharp-crested weirs can be extended to the channel.

Assuming that the area of flow in the shaded portion *A1* is the additional flow area being substituted by the flow area that is represented by the shaded portions *A2*. This assumption is similar to Stouts and Sutro weirs [13, 15, 16] wherein the infinite crest width was substituted with the rectangular weir to match the flows between the two weirs.

For the first time, it was found that even in rectangular flume it is possible to get a near linear depth-discharge relationship. The flume can be considered to be similar to that of a compound proportional weir wherein the flow with desired head-discharge relationship is valid only in the complementary weir portion and in case of base weir flow the regular discharge equation is to valid. Extending the same argument for flumes, beyond a small portion of depth of flow from the bed of the channel the near linear depth-discharge equation will be valid. The initial depth where the proposed equation being not valid is ***threshold depth***. For flow in that portion of the threshold depth, normal rectangular flume depth-discharge equation is valid. An algebraic optimization procedure on the lines of that developed by Shesha Prakash is derived and the same is proposed in the present research work.

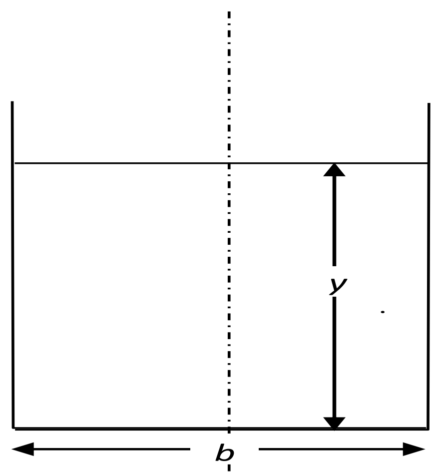

Fig 4: Definition Sketch of Rectangular Flume

Consider the flow through the rectangular channel.

$\text{cross} - \text{sectional area of flow} = \ a = b \times y$

$\text{wetted perimeter} = \ p = b + 2y$

Hydraulic mean radius $= R = \frac{a}{p}$

$$= \frac{by}{(b+2y)}$$

From Chezy′sequation, Discharge $= q = aC\sqrt{RS}$

$$q = C\sqrt{S}\frac{a^{\frac{3}{2}}}{p^{\frac{1}{2}}} = C\sqrt{S}\frac{(by)^{\frac{3}{2}}}{(b+2y)^{\frac{1}{2}}}$$

$$q = C\sqrt{S}\frac{b^{\frac{3}{2}}\left(b\frac{y}{b}\right)^{\frac{3}{2}}}{b^{\frac{1}{2}}\left(1+2\frac{y}{b}\right)^{\frac{1}{2}}} = C\sqrt{S}\frac{b^{3}\left(\frac{y}{b}\right)^{\frac{3}{2}}}{b^{\frac{1}{2}}\left(1+2\frac{y}{b}\right)^{\frac{1}{2}}}$$

Non − dimensionalising with $Y = \left(\frac{y}{b}\right)$;

$$q = C\sqrt{S}\frac{b^{\frac{5}{2}}Y^{\frac{3}{2}}}{\sqrt{1+2Y}};$$

$$Q = \frac{q}{C\sqrt{S}b^{\frac{5}{2}}} = \frac{Y^{\frac{3}{2}}}{\sqrt{(1+2Y)}}$$

4. OPTIMIZATION PROCEDURE

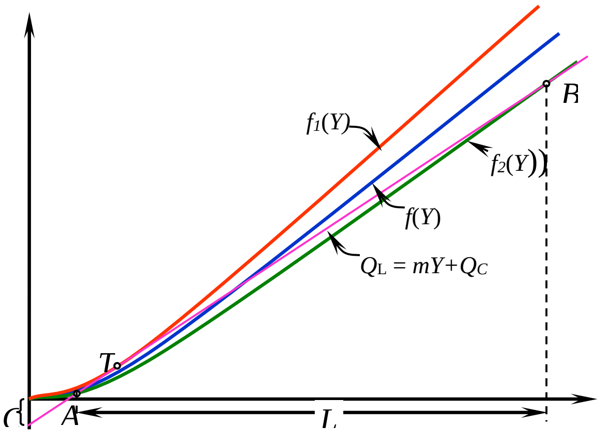

Fig 5: Optimization Procedure

To obtain a maximum straight-line between two bound curves for an ever increasing function between two parameters.

Let

$$\boldsymbol{Q_L = mY + Q_c} \ \textbf{... (4.01)}$$

Be the proposed optimal linear head-discharge relationship (where, *m* is the constant of proportionality and *Qc* is the discharge intercept) to substitute the theoretical head-discharge relationship.

$$\boldsymbol{Q = f(Y)} \ \textbf{.... (4.02)}$$

In a certain range. Letting $K_u = (1 + \frac{E}{100})$ and $K_d = (1 - \frac{E}{100})$ where E is the prefixed maximum permissible relative deviation of the proposed linear function and the theoretical head-discharge function. These defined two explicit curves $f_1(Y)$ and $f_2(Y)$ forming the lower and upper bounds for the linear function as shown in figure 5.

$$\boldsymbol{f_1(Y) = K_u f(Y)} \textbf{... (4.03)}$$
$$\boldsymbol{f_2(Y) = K_d f(Y)} \textbf{ ... (4.04)}$$

Following from 4.04, the solution can be obtained using the algorithm below.

- **Obtain the point of inflection on the curve for the discharge head**
- **Run linear regression between the two curves at positive and negative errors**
- **Obtain a common tangent to the straight-line section of the curves**
- **The limits of the tangent segment will be the linear operation region for the weir**

Step 1: Find the point of inflection of the curve $Q = f(Y)$ given by Eq.(4.02)

i.e $\boldsymbol{Q'' = f''(Y) = 0}$

This procedure is valid only when there is one point of inflection and the function is continuously increasing in $0 \leq H \leq \infty$ which is always true for any discharge-head function. No real roots existing, hence no point of inflection. This means *H* is a continuously increasing curve.

Step 2: Now the objective is to find the maximum straight-line relationship within the error bound curves, Eqs. (4.03) and (4.04).

The flow parameters for flow through rectangular flume such as starting point *(A)* of the near linear flow equation (can be the threshold depth, below which the proposed equation is not valid), Ending point *(B)*, beyond which the proposed equation is invalid and the linear flow equation itself in the form of Eq. (4.01) (for which m and *Qc* are to be evaluated).

Following are the two considerations for an existing rectangular channel:

1. Pre-fix the threshold depth, only beyond which the flow equation will be valid. It can be fixed in terms of percentage of total non-dimensional depth with which the starting point *A* will be known and be can be computed.

2. With the existing channel height being known point *B* could be pre fixed from which point *A* can be computed.
3. The range in which the linear flow equation is valid will be between *A* and *B*

This straight line to be maximum, it is proposed to be tangential to upper bound curve *f1(Y)* at point *T*. Let us assume that with a slight variation of '*m*' we get a straight line longer than the one given by given by Eq.(4.01).

$$\boldsymbol{Q_L = (m \pm \Delta m)Y + Qc \ldots} \textbf{ (4.05)}$$

But $(m + \Delta m)$ shifts the line beyond the boundary $f_1(Y)$ at T and $(m - \Delta m)$ for $f_2(Y)$ the end point will be shortened because the straight line intersects the bottom curve earlier due to the reduction in slope. Hence, the value of m is the optimum which yields the straight line of maximum length.
Now let us consider a small variation in "*Qc*" as $(Q_c \pm \Delta Q_c)$ with which we get a longer straight line than the one given by Eq. (1.01).

i.e $\boldsymbol{Q_L = mY + (Q_c \pm \Delta Q_c)}$ **….(4.06)**

But $(Q_c + \Delta Q_c)$ shifts the line beyond the boundary $f_1(Y)$ at *T* and $(Q_c - \Delta Q_c)$ for $f_2(Y)$ at B. Hence the value of Q_c is the optimum value along with m which yields the straight line of maximum length. Thus Eq. (4.01), gives us the straight line of maximum length hence the maximum linearity range.
To obtain the flow parameters for the present model,

Step 3: The non-dimensional flow equation through the rectangular channel is given by

$$\boldsymbol{Q = \frac{Y^{3/2}}{\sqrt{2Y+1}} \ldots} \textbf{ (4.07)}$$

Differentiating, w.r.t. *Y*

$$\boldsymbol{Q' = \frac{dQ}{dy} = \frac{\sqrt{Y}(4Y+3)}{2(2Y+1)^{3/2}} \ldots} \textbf{ (4.08)}$$

$$E = \frac{e}{100}$$

$=$ the maximum permissible and prefixed percentage error

<u>At '*A*' starting point</u>

Point *A* can be found as explained in the previous steps.

$$\boldsymbol{(Q_L)_A = mY_A + Q_c \ldots} \textbf{ (4.09a)}$$

$$\boldsymbol{(Q_D)_A = (1 - E)\frac{Y_A^{3/2}}{\sqrt{2Y_A+1}} \ldots} \textbf{ (4.10a)}$$

$$\boldsymbol{(Q_L)_A = (Q_D)_A \; or \; mY_A + Q_c = (1 - E)\frac{Y_A^{3/2}}{\sqrt{2Y_A+1}}} \textbf{ (4.11a)}$$

<u>At '*B*' starting point</u>

Point *B* can be found as explained in the previous steps.

$$\boldsymbol{(Q_L)_B = mY_B + Q_c \ldots} \textbf{(4.09b)}$$

$$\boldsymbol{(Q_D)_B = (1 - E)\frac{Y_B^{3/2}}{\sqrt{2Y_B+1}} \ldots} \textbf{.(4.10b)}$$

$$\boldsymbol{(Q_L)_B = (Q_D)_B \; or \; mY_B + Q_c = (1 - E)\frac{Y_B^{3/2}}{\sqrt{2Y_B+1}}} \textbf{ (4.11b)}$$

At '*T*' Tangent to upper curve

$$\boldsymbol{(Q_L)_T = mY_T + Q_c \ldots} \textbf{(4.12)}$$

$$\boldsymbol{(Q_U)_T = (1 + E)\frac{Y_T^{3/2}}{\sqrt{2Y_T+1}} \ldots} \textbf{ (4.13)}$$

Differentiating, Eqs. (4.12) and (4.13) w.r.t. *Y* and from Eq. (4.08)

$$\boldsymbol{\left(\frac{dQ_L}{dY}\right)_T = m = (1 + E)\frac{\sqrt{Y_T}(4Y_T+3)}{2(2Y_T+1)^{3/2}} \ldots} \textbf{ (4.14)}$$

Substituting for '*m*' in Eqs. (4.12) and (4.13)

$$\boldsymbol{(1 + E)\frac{Y_T^{3/2}(4Y_T+3)}{2(2Y_T+1)^{3/2}} + Q_c = (1 + E)\frac{Y_T^{3/2}}{\sqrt{2Y_T+1}}} \textbf{ ..(4.15)}$$

Stage 1 (By knowing Y_A)

Substituting for '*m*' in Eq. 4.11a

$$(1 + E)\frac{\sqrt{Y_T}(4Y_T + 3)}{2(2Y_T + 1)^{3/2}}Y_A + Q_c = (1 - E)\frac{Y_A^{3/2}}{\sqrt{2Y_A + 1}}$$

$$\boldsymbol{Q_c = (1 - E)\frac{Y_A^{3/2}}{\sqrt{2Y_A+1}} - (1 + E)\frac{\sqrt{Y_T}(4Y_T+3)}{2(2Y_T+1)^{3/2}}Y_A} \textbf{ (4.16a)}$$

Substituting for Q_c, from Eq. (4.16a) into (4.15) we get,

$$\boldsymbol{(1 + E)\frac{Y_T^{\frac{3}{2}}(4Y_T+3)}{2(2Y_T+1)^{\frac{3}{2}}} + (1 - E)\frac{Y_A^{\frac{3}{2}}}{\sqrt{2Y_A+1}} - (1 + E)\frac{\sqrt{Y_T}(4Y_T+3)}{2(2Y_T+1)^{\frac{3}{2}}}Y_A - (1 + E)\frac{Y_T^{\frac{3}{2}}}{\sqrt{2Y_T+1}} = 0} \textbf{ (4.17a)}$$

Stage 2 (By knowing Y_B)

Substituting for '*m*' in Eq. 4.11b

$$\boldsymbol{(1 + E)\frac{\sqrt{Y_T}(4Y_T + 3)}{2(2Y_T + 1)^{3/2}}Y_B + Q_c = (1 - E)\frac{Y_B^{3/2}}{\sqrt{2Y_B + 1}}}$$

$$Q_c = (1-E)\frac{Y_B^{3/2}}{\sqrt{2Y_B+1}} - (1+E)\frac{\sqrt{Y_T}(4Y_T+3)}{2(2Y_T+1)^{3/2}}Y_B$$

(4.16b)

Substituting for Q_c, from Eq. (4.16b) into (4.15) we get,

$$(1+E)\frac{Y_T^{\frac{3}{2}}(4Y_T+3)}{2(2Y_T+1)^{\frac{3}{2}}} + (1-E)\frac{Y_A^{\frac{3}{2}}}{\sqrt{2Y_A+1}} - (1+E)\frac{\sqrt{Y_T}(4Y_T+3)}{2(2Y_T+1)^{\frac{3}{2}}}Y_A - (1+E)\frac{Y_T^{\frac{3}{2}}}{\sqrt{2Y_T+1}} = 0$$

(4.17b)

Solving for Y_T from Eqs. (4.17a and 4.17b) and substituting in Eqs. (4.14) and (4.16), we can evaluate *m* and *Qc* to get the linear depth-discharge equation for flow in rectangular channel. This equation is valid for flow depths Y_A to Y_B.

$$(Q_L)_B = (Q_D)_B \text{ or } mY_B + Q_c = (1-E)\frac{Y_B^{3/2}}{\sqrt{2Y_B+1}} \quad \textbf{(4.18)}$$ and

$$(Q_L)_A = (Q_D)_A \text{ or } mY_A + Q_c = (1-E)\frac{Y_B^{3/2}}{\sqrt{2Y_B+1}} \quad \textbf{(4.19)}$$

$$Q_L = mY + Q_c \text{ ......} YA \leq Y \leq YB \text{.....} \quad \textbf{(4.20)}$$

The above is the developed theorem to obtain the linear relationship, figure 6 depicts the plot showing values from known *A* to *B*, figure 7 depicts the plot showing values from known *B* to *A*.

4.1 Validation through Graphical Method

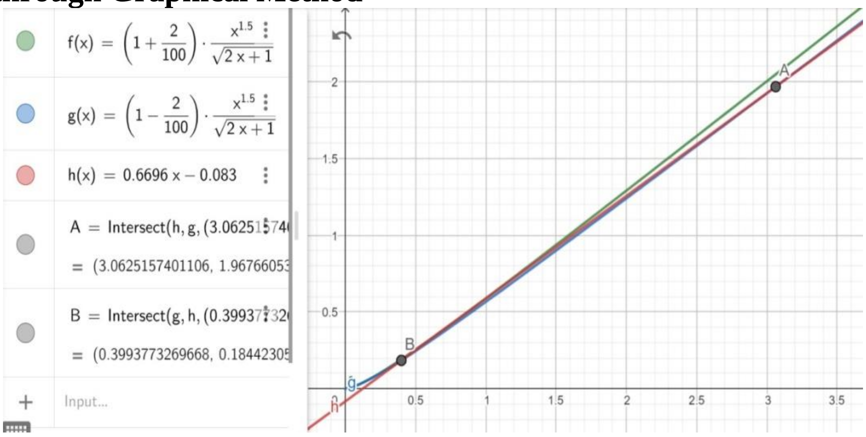

Fig 6: Plot showing values from known *A* to *B*

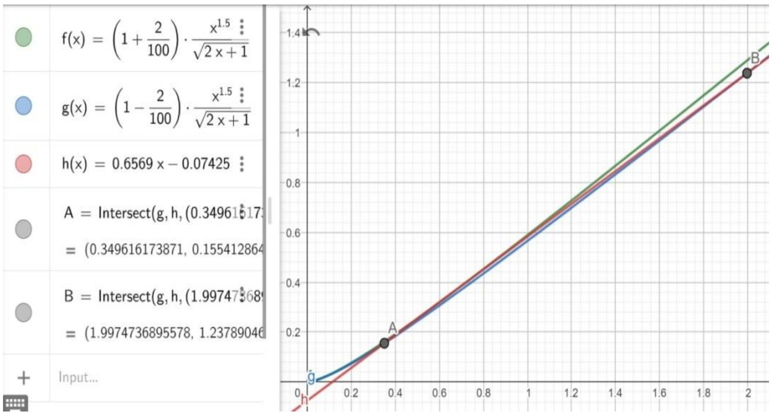

Fig 7: Plot showing values from known *B* to *A*

4.2 Analysis

The below 2 tables will provide the flow parameters based on the two conditions, viz. known threshold depth (*A*) and known channel depth (*B*)

Table 1: Flow parameters for known threshold depth (Y_A)

Y_A	Y_B	m	Q_c
0.1	0.281129	0.48797	-0.02051
0.2	0.596683	0.579696	-0.04186
0.3	1.372236	0.640304	-0.06479
0.4	3.063444	0.669643	-0.08307
0.5	13.45666	0.68779	-0.0989
0.6	Infinity	0.698877	-0.11225
0.7	Infinity	0.706657	-0.12418
0.8	Infinity	0.711934	-0.13466
0.9	Infinity	0.71541	-0.14382
1	Infinity	0.718002	-0.1522

Table 2: Flow parameters for known End point (Height of the channel or known flow-depth ($\boldsymbol{Y_B}$))

Y_A	Y_B	m	Q_c
2	0.344804	0.656933	-0.07425
1.9	0.346529	0.655276	-0.07354
1.8	0.333	0.653172	-0.07225
1.7	0.331385	0.65084	-0.07087
1.6	0.324363	0.648226	-0.06937
1.5	0.316533	0.645276	-0.06773
1.4	0.3	0.641925	-0.06592
1.3	0.3	0.638085	-0.06393
1.2	0.297	0.633645	-0.06173
1.1	0.272751	0.628453	-0.05927

Where $Q_L = mY + Q_c$……… $(Y_A \leq Y \leq Y_B)$

5. CONCLUSION

The Flow measurement has taken an all-important and significant stage in the present day of severe water shortage. Measurement of flow with least interference is the best suited device for minimizing head losses. Sharp and Broad crested weirs, which are the most popular and accurate flow measuring structures have a coefficient of discharge of about 0.6 which accounts for about 40% head loss relative to the theoretical discharge. Further the accuracy of measurement and computations are also under scanner and questionable. The accuracy of flow measurement through Venturi flume and Standing wave flume are depending on the accuracy of fabrication as per the design. Hence, the proposed linear discharge-depth relationship in rectangular channel, with its geometrically simple device, with least flow interference and near accurate measurement can be best alternative. In addition, no computations are required for measurement of discharges as it can be directly read on a scale which prompts even illiterate farmers to install such devices.

The simple linear discharge-depth equation that is proposed deviates less than 2% with the theoretical discharge. In addition, the threshold depth, beyond which the proposed linear depth-discharge relationship is valid can be suitably designed as per the requirements. The linear relationship can be valid for either known or prefixed threshold depth or by knowing the height of the channel, the threshold depth can be fixed.

Further, the near linear depth-discharge relationship is valid from $Y_A(b)$ to $Y_B(b)$, within a deviation of ±2 percent error, where b is the half base width of the channel. The proposed linear equation is given by $q_L = m\,(y - Q_c)C\sqrt{S}b^{3/2}$, where q_L is the discharge in the channel, y is the flow depth, C is Chezzy's Constant and S is the channel bed slope. Chezzy's C can also be substituted by Manning's n by the simple equation $C = \frac{1}{n}R^{1/6}$, where R the Hydraulic mean radius could be computed as $R = \frac{a}{P}$.

a being the cross-sectional area of flow and P the wetted perimeter.

Further, as the discharge is linearly varying with the depth of flow, the discharges for various depths could be computed and the converted values of discharge printed on a linear scale could be printed on the piezometer attached in required units as litres per second or minute. The least count of the measurement decreases, increasing the sensitivity. The proposed measurement through the existing rectangular channels will be highly useful in practice in Irrigation, chemical and Hydraulic engineering, by providing least interference in flow. The experimental verification and the design characteristics will be done as future scope of the present research work.

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