Linear Maps that are (*g*,*h*)-Derivations or (*g*,*t*,*h*)-Ternary Derivations at a Point on *-Module Extension Banach Algebras

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Abstract

We introduce a *-module extension Banach algebras to generalized the results of Essaleh and Peralta. Precisely, let g,t and h are bounded homomorphism maps on an unital *-module extension Banach algebra, if a bounded linear map D on an unital *-module extension Banach algebra is (g,t,h)-ternary derivation at the unit element, then the next statements are hold:

1) D is (g,h)-generalized derivation;

2) D is *-(g,h)-derivation and (g,t,h)-triple (ternary) derivation, whenever D(1,0)=(0,0);

3) D is (g,t,h)-ternary derivation.

In addition, we prove that a bounded linear map on *-module extension Banach algebra which is (g,h)-derivation or (g,t,h)-ternary derivation at the zero element is (g,h)-generalized derivation.

Keywords: **-module extension Banach algebras, (g,h)-derivations, triple (ternary) derivations, (g,h)-generalized derivation, derivable mapping at a point.*

1. Introduction

One of the studied types of operators that received the greatest attention is derivations on Banach algebras. Modern studies are attentive to the problem to find weaker conditions for the description of these maps. Derivation at a certain point considers one of the studies of the fruitful maps. Let A be a Banach algebra, and let X be a Banach A-bimodule. The l^1- direct sum of a Banach algebra A and a nonzero Banach A-bimodule X, is the module extension with the algebraic operations which are defined as follows: $(ş, ŋ) + (ŗ, ɱ) = (ş + ŗ, ŋ + ɱ)$, $ŗ(ş, ŋ) = (ŗ\, ş, ŗ\, ŋ), (ş, ŋ)ŗ = (ş\, ŗ, ŋ\, ŗ), (ş, ŋ)(ŗ, ɱ) = (ş\, ŗ, ş\, ɱ + ŋ\, ŗ)$, for all $ş, ŗ \in A, ŋ, ɱ \in X$. And it is obvious that $A \oplus X$ is a Banach algebra with the following norm; $\| (ş, ŋ) \| = \| ş \| + \| ŋ \|$, for all $ş \in A$, $ŋ \in X$. There are many researchers who studied this type of Banach algebras from different sides; see for example [19, 26]. A $*$-module extension Banach algebra is a module extension Banach algebra $A \oplus X$ with an involution mapping $* : A \oplus X \to A \oplus X$, denoted by $*$-$A \oplus X$, such that the mapping $*$

satisfying the properties: $\big((ş,ŋ)+(ŗ,m̧)\big)^{*}=(ş,ŋ)^{*}+(ŗ,m̧)^{*}$, $(1,0)^{*}=(1,0)$, $\big((ş,ŋ)\,(ŗ,m̧)\big)^{*}=(ŗ,m̧)^{*}(ş,ŋ)^{*}, ((ş,ŋ)^{*})^{*}=(ş,ŋ)$, for all $(ş,ŋ)$, $(ŗ,m̧)$ in $*$-$A\bigoplus X$, and $(1,0)$ is the unit element of $*$-$A\bigoplus X$. Let U be a $(*$-$A\bigoplus X)$-bimodule. A linear map $D: *$-$A\bigoplus X\rightarrow U$ is called a (g,h)-derivation at a point $(ç,z̧)$ in $*$-$A\bigoplus X$ (is also say that derivable at $(ç,z̧)$) where g,h: $*$-$A\bigoplus X\rightarrow U$ are linear maps, if the identity: $D\big((ş,ŋ)(ŗ,m̧)\big)=D(ş,ŋ)\,h(ŗ,m̧)+g(ş,ŋ)\,D(ŗ,m̧)$, holds for all $(ş,ŋ)$, $(ŗ,m̧)$ in $*$-$A\bigoplus X$ such that $(ş,ŋ)\,(ŗ,m̧)=(ç,z̧)$ [21]. There exists a linear map is (g,h)-derivation at a certain point, but not necessary to be (g,h)-derivation. Following to ([2], [17] and [21]), we will say that a linear map $D: *$-$A\bigoplus X\rightarrow U$ is said to be a (g,h)-generalized derivation, where $g,h: *$-$A\bigoplus X\rightarrow U$ are linear maps, if there exists ζ in U^{**} satisfying: $D\big((ş,ŋ)(ŗ,m̧)\big)=D(ş,ŋ)\,h(ŗ,m̧)+g(ş,ŋ)\;D(ŗ,m̧)-g(ş,ŋ)\;D(\zeta)\;h(ŗ,m̧)$, for all $(ş,ŋ),(ŗ,m̧)$ in $*$-$A\bigoplus X$. It is well known that each (g,h)-derivation is a (g,h)-generalized derivation, but the converse is not true. This idea is more helpful whenever describing (generalized) derivations of the annihilation of specific products of perpendicular elements [3, Theorem 2.11]. A $*$-module extension Banach algebra $*$-$A\bigoplus X$ is a JB*-ternary with respect to the ternary product defined by: $\{(ş,ŋ),(ŗ,m̧),(ç,z̧)\}=\frac{1}{2}\big((ş,ŋ)\,(ŗ,m̧)^{*}\,(ç,z̧)+(ç,z̧)\,(ŗ,m̧)^{*}\,(ş,ŋ)\big)$. This is the normal ternary product that appears in ([4], [9], [10], [11], [20] and [25]). A linear map D: $*$-$A\bigoplus X\rightarrow U$ is called (g,t,h)-ternary derivation where g,t,h:$*$-$A\bigoplus X\rightarrow U$ are linear maps, if the identity: $D\,\{(ş,ŋ),(ŗ,m̧),(ç,z̧)\}=\{D(ş,ŋ),\,t(ŗ,m̧),h\,(ç,z̧)\}+\{g(ş,ŋ),D(ŗ,m̧),h(ç,z̧)\}+\{g(ş,ŋ),t(ŗ,m̧),D\,(ç,z̧)\}$, holds for every $(ş,ŋ),(ŗ,m̧)$, $(ç,z̧)$ in $*$-$A\oplus X$ [6]. According to [7], suppose $*$-$A\oplus X$ and $*$-$B\oplus U$ are $*$-module extension Banach algebras, and let $D: *$-$A\oplus X\rightarrow *$-$B\bigoplus U$ be a linear map. Assume that a linear map $D^{\#}: *$-$A\oplus X\rightarrow *$-$B\oplus U$ defined by $D^{\#}(ş,ŋ)=D((ş,ŋ)^{*})^{*}$, for all $(ş,ŋ)$ in $*$-$A\oplus X$. We say that D is a symmetric when $D^{\#}=D$. A $*$-(g,h)-derivation D on $*$-$A\oplus X$ is (g,h)-derivation on $*$-$A\oplus X$ and a symmetric map (i.e., $D\big((ş,ŋ)\big)^{*}=D((ş,ŋ)^{*})$, for all $(ş,ŋ)$ in $*$-$A\oplus X$). Examples of (g,h)-derivations on a $*$-$A\oplus X$, we will consider $g,t,h: *$-$A\oplus X\rightarrow *$-$A\oplus X$ are homomorphism maps and we can fix $(ş,ŋ)$ in $*$-$A\oplus X$ and define a linear map $D_{(ş,ŋ)}: *$-$A\oplus X\rightarrow *$-$A\oplus X$ by $D_{(ş,ŋ)}(ŗ,m̧)=[(ş,ŋ),(ŗ,m̧)]=(ş,ŋ)\,t(ŗ,m̧)-g(ŗ,m̧)\,(ş,ŋ)$. The fact that all $*$-(g,h)-derivation on $*$-$A\oplus X$ is (g,t,h)-ternary derivation as described above. Furthermore, there exist (g,h)- derivations on $*$-$A\oplus X$ that are not (g,t,h)-ternary derivations see ([12, Proof of Lemma 1] and [5, Comments after Lemma 3). According to [6], let D,g,t and h :$*$-$A\oplus X\rightarrow *$-$A\oplus X$ be linear maps, and let (a_1,a_2) be an element in $*$-$A\oplus X$. We will say that D is (g,t,h)-ternary derivation at (a_1,a_2) if $(a_1,a_2)=\{(ş,ŋ),(ŗ,m̧),(ç,z̧)\}$ in $*$-$A\oplus X$ implies that $D(a_1,a_2)=\{\,D(ş,ŋ),\;t(ŗ,m̧),\;h\,(ç,z̧)\,\}+\{\,g(ş,ŋ),\;D(ŗ,m̧),\;h\,(ç,z̧)\,\}+\{g(ş,ŋ),t(ŗ,m̧),D(ç,z̧)\}$. In the literary, a linear map D on $*$-$A\oplus X$ which is (g,t,h)-ternary derivation at (a_1,a_2) is also called (g,t,h)-ternary derivable at (a_1,a_2). We recall that, for a C^*- algebra A, suppose D is a bounded linear map defined on unital $*$-$A \oplus X$, and let g, t and h be continuous homomorphisms on unital $*$-$A \oplus X$. The self-adjoint set of $*$-$A \oplus X$ will be represented by the symbol $(* - A \oplus X)_{sa}$. In [6] Essaleh and Peralta proved that if a continuous linear map D defined on unital A is ternary derivation at the unit element of A, then D is generalized derivation. When they suppose that $D(1) = 0$, then D is $*$-derivation and also ternary derivation. In this paper, we generalize the previous results by proving that a bounded linear map D on unital $*$-$A \oplus X$ which is (g, t, h)-ternary derivation at the unit (1,0) is (g, h)-generalized derivation. When we added $D(1,0) = (0,0)$, then D is $*$-(g, h)-derivation and hence (g, t, h)-ternary derivation (see Theorem 2.1.5 and Proposition 2.1.7). Furthermore, we prove that if a continuous linear map D on unital $*$-$A \oplus X$ is (g, t, h)-ternary derivation at (1,0), then D is (g, t, h)-ternary derivation (see Corollary 2.1.8). Finally, for a (g, t, h)-ternary derivation at zero on $*$-$A \oplus X$. We shall extension of ([6, Theorem 2.9] and [14, Theorem 4]) by using unital $*$-$A \oplus X$. we prove that if a bounded linear map D on $*$-$A \oplus X$ is (g, h)-derivation or (g, t, h)- ternary derivation at zero, then D is (g, h)-generalized derivation (see Theorem 2.2.3).

2. (g,t,h)-Ternary Derivations at Stable Point of a *-Module Extension Banach Algebra

Throughout this section, we will focus on linear maps between $*$-module extension Banach algebras that are (g, t, h)-ternary derivations at a stable point. There exist two salient elements that should be taken into consideration at the beginning of any study, we mean the unit and zero element are those two elements of $*$-module extension Banach algebra. Later, we will show that (g, t, h)-ternary derivations at unit or at zero element between $*$-module extension Banach algebras are a fundamental connection to (g, h)-generalized derivations. We need the following proposition to prove our results.

Note that, the $*$-module extension Banach algebra $*$-$B \oplus U$ in the following proposition corresponds with $*$-$A \oplus X$ or with any $*$-module extension Banach algebra containing $*$-$A \oplus X$ like a $*$-submodule extension Banach algebra with the same unit.

Proposition 2.1

Let D be bounded linear map from unital $*$-$A \oplus X$ into a Banach $(*$-$A \oplus X)$-bimodule $*$-$B \oplus U$, where g and h from $* - A \oplus X$ into $* - B \oplus U$ are bounded homomorphisms. The next arguments are equivalent:

1) D is (g, h)-generalized derivation;
2) $g(\text{ş}, \text{ŋ}) D(\text{ŗ}, \text{ɱ}) h(\text{ç}, \text{ʐ}) = (0,0)$, whenever $(\text{ş}, \text{ŋ})\ (\text{ŗ}, \text{ɱ}) = (\text{ŗ}, \text{ɱ})\ (\text{ç}, \text{ʐ}) = (0,0)$ in $*$-$A \oplus X$;
3) $g(\text{ş}, \text{ŋ}) D(\text{ŗ}, \text{ɱ}) h(\text{ç}, \text{ʐ}) = (0,0)$, whenever $(\text{ş}, \text{ŋ})\ (\text{ŗ}, \text{ɱ}) = (\text{ŗ}, \text{ɱ})\ (\text{ç}, \text{ʐ}) = (0,0)$ in $(* - A \oplus X)_{sa}$. Furthermore, each argument in (1) - (3) is equivalent to any one of the following:
4) $g(\text{ş}, \text{ŋ})\ D(\text{ŗ}, \text{ɱ})\ h(\text{ç}, \text{ʐ}) + h(\text{ç}, \text{ʐ})\ D(\text{ŗ}, \text{ɱ})\ g(\text{ş}, \text{ŋ}) = (0,0)$, whenever $(\text{ş}, \text{ŋ})\ (\text{ŗ}, \text{ɱ}) = (\text{ŗ}, \text{ɱ})(\text{ç}, \text{ʐ}) = (0,0)$ in $(* - A \oplus X)_{sa}$;
5) $g(\text{ş}, \text{ŋ})\ D(\text{ŗ}, \text{ɱ})\ h(\text{ş}, \text{ŋ}) = (0,0)$, whenever $(\text{ş}, \text{ŋ})\ (\text{ŗ}, \text{ɱ}) = (0,0)$ in $(* - A \oplus X)_{sa}$;
6) For every $(\text{ŗ}, \text{ɱ})$ in $(* - A \oplus X)_{sa}$ we get $g\big((1,0) - R(\text{ŗ}, \text{ɱ})\big)\ D(\text{ŗ}, \text{ɱ})\ h\big((1,0) - R(\text{ŗ}, \text{ɱ})\big) = (0,0)$ in $(* - B \oplus U)^{**}$, where

$R(r, m)$ symbolizes the range projection of $(\mathrm{ṛ}, \mathrm{ṃ})$ in $(* - A \oplus X)^{**}$.

Proof. The proof of the first part is comparable to that of ([3, Proposition 2.8] and [17, Proposition 1.1]). For the proof of the second part, (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) are clear. Now, we will prove (5) $\Longrightarrow$ (6) Let $g(\mathrm{ş}, \mathrm{ņ})\ D(\mathrm{ṛ}, \mathrm{ṃ})\ h(\mathrm{ş}, \mathrm{ņ}) = (0,0)$, with $(\mathrm{ş}, \mathrm{ņ})\ (\mathrm{ṛ}, \mathrm{ṃ}) = (0,0)$ in $(* - A \oplus X)_{sa}$. Assume that $A_{\mathrm{ṛ}}$ is a commutative C^*-subalgebra of A which is generated by $(\mathrm{ṛ})$. From the Gelfand theory, we have $A_{\mathrm{ṛ}} \cong C_\circ(\sigma(\mathrm{ṛ}))$, such that $C_\circ(\sigma(\mathrm{ṛ}))$ denotes C^*-algebra of each bounded functions on $\sigma(\mathrm{ṛ})$ finish at zero, and $\sigma(\mathrm{ṛ}) \subseteq [-\| \mathrm{ṛ} \|, \| \mathrm{ṛ} \|]$ refers to the spectrum of ṛ. For every natural α, let a projection $(a_\alpha, x) \in (* - A \oplus X)^{**}_{(\mathrm{ṛ},\mathrm{ṃ})} \subseteq (* - A \oplus X)^{**}$ according to the function of the set $([-\| \mathrm{ṛ} \|, -\frac{1}{\alpha}] \cup [\frac{1}{\alpha}, \| \mathrm{ṛ} \|]) \cap \sigma(\mathrm{ṛ})$. Let us choose a function $(\mathrm{ṛ}_\alpha, \mathrm{ṃ}) \in (* - A \oplus X)_{(\mathrm{ṛ},\mathrm{ṃ})}$, with $(\mathrm{ṛ}_\alpha, \mathrm{ṃ})\ (a_\alpha, x) = (a_\alpha, x)\ (\mathrm{ṛ}_\alpha, \mathrm{ṃ}) = (\mathrm{ṛ}_\alpha, \mathrm{ṃ}) = (\mathrm{ṛ}_\alpha, \mathrm{ṃ})^*$, and $\| (\mathrm{ṛ}_\alpha, \mathrm{ṃ}) - (\mathrm{ṛ}, \mathrm{ṃ}) \| \leq \frac{1}{\alpha}$. It is clear that a sequence (a_α, x) is convergent to $R(\mathrm{ṛ}, \mathrm{ṃ})$ in the $strong^*$-topology of $(* - A \oplus X)^{**}$ see [23, Ş1.8]. And let us pick $(b, -x) \in ((1,0) - (a_\alpha, x))\ (* - A \oplus X)^{**}\ ((1,0) - (a_\alpha, x)) \cap (* - A \oplus X)_{sa}$. Since $(\mathrm{ṛ}_\alpha, \mathrm{ṃ})\ (b, -x) = (0,0)$. By the hypothesis, we have that $g(b, -x)\ D(\mathrm{ṛ}_\alpha, \mathrm{ṃ})\ h(b, -x) = (0,0)$. Moreover, from ([24, Definition III.6.19] and [1]), a projection (a_α, x) be a closed in $(* - A \oplus X)^{**}_{(\mathrm{ṛ},\mathrm{ṃ})} \subseteq (* - A \oplus X)^{**}$. This implies that $1 - a_\alpha$ is open projection in A^{**}. So, there is increasing net $(b_\beta) \in (1 - a_\alpha)A^{**}\ (1 - a_\alpha) \cap A$, with $0 \leq (b_\beta) \leq 1 - a_\alpha$, and (b_β) is convergent to $(1 - a_\alpha)$ in the $weak^*$- topology of A^{**} see ([1], [8] and [22, Proposition 3.11.9]). Since $0 \leq \left((1 - a_\alpha) - (b_\beta)\right)^2 \leq (1 - a_\alpha) - (b_\beta) \leq (1 - a_\alpha)$. So, (b_β) is convergent to $(1 - a_\alpha)$ in the $strong^*$- topology of A^{**}. We have that $g(b_\beta, -x)\ D(\mathrm{ṛ}_\alpha, \mathrm{ṃ})\ h(b_\beta, -x) = (0,0)$, for all β. From [23, Proposition 1.8.12], we have that the product of A is together $strong^*$-continuous. Thus, $g((1,0) - (a_\alpha, x))\ D(\mathrm{ṛ}_\alpha, \mathrm{ṃ})\ h((1,0) - (a_\alpha, x)) = (0,0)$, for each natural α. Since $(1,0) - (a_\alpha, x)$ converges to $(1,0) - R(\mathrm{ṛ}, \mathrm{ṃ})$ in the $strong^*$-topology and $D(\mathrm{ṛ}_\alpha, \mathrm{ṃ})$ converges to $D(\mathrm{ṛ}, \mathrm{ṃ})$ in norm. This implies that $g((1,0) - R(\mathrm{ṛ}, \mathrm{ṃ}))\ D(\mathrm{ṛ}, \mathrm{ṃ})\ h((1,0) - R(\mathrm{ṛ}, \mathrm{ṃ})) = (0,0)$ in $(* - A \oplus X)^{**}$.

(6) $\Longrightarrow$ (3) Assume that $(\mathrm{ş}, \mathrm{ņ}), (\mathrm{ṛ}, \mathrm{ṃ}), (\mathrm{ç}, \mathrm{ẓ})$ in $(* - A \oplus X)_{sa}$ such that $(\mathrm{ş}, \mathrm{ņ})\ (\mathrm{ṛ}, \mathrm{ṃ}) = (\mathrm{ṛ}, \mathrm{ṃ})\ (\mathrm{ç}, \mathrm{ẓ}) = (0,0)$. We observe that $g(\mathrm{ş}, \mathrm{ņ}) = g(\mathrm{ş}, \mathrm{ņ})\ g((1,0) - R(\mathrm{ṛ}, \mathrm{ṃ}))$, and $h(\mathrm{ç}, \mathrm{ẓ}) = h((1,0) - R(\mathrm{ṛ}, \mathrm{ṃ}))\ h(\mathrm{ç}, \mathrm{ẓ})$. So, we conclude that $g(\mathrm{ş}, \mathrm{ņ})\ D(\mathrm{ṛ}, \mathrm{ṃ})\ h(\mathrm{ç}, \mathrm{ẓ}) = g(\mathrm{ş}, \mathrm{ņ})\ g((1,0) - R(\mathrm{ṛ}, \mathrm{ṃ}))\ D(\mathrm{ṛ}, \mathrm{ṃ})\ h((1,0) - R(\mathrm{ṛ}, \mathrm{ṃ}))\ h(\mathrm{ç}, \mathrm{ẓ}) = (0,0)$, which complete the proof.

2.1 (g, t, h)-Ternary Derivations of a $*$-Module Extension Banach Algebra at the Unit Element

In this subsection, $*$-$A \oplus X$ will denote a $*$-submodule extension Banach algebra of unital $*$-module extension Banach algebra $*$-$B \oplus U$ which contains the unit element of $*$-$B \oplus U$, and we will assume that D from $*$- $A \oplus X$ into $*$- $B \oplus U$ is a continuous linear map, also g, t and h from $*$- $A \oplus X$ into $*$- $B \oplus U$ are continuous homomorphisms.

Proposition 2.1.1

Assume that $A \oplus X$ is a module extension Banach algebra with unit $(1,0)$, and let $B \oplus U$ be a unital Banach $A \oplus X$ -bimodule. Suppose D, g and $h : A \oplus X \rightarrow B \oplus U$ are continuous linear maps, then D is (g,h)-derivation at the unit element, if and only if D is a Jorden (g,h)-derivation.

Proof. The proof is like to that of [18, Theorem 2.1 or Corollary 2.3].

Proposition 2.1.2

Suppose $A \oplus X$ is a symmetric amenability module extension Banach algebra, and let $B \oplus U$ be a Banach $A \oplus X$-bimodule, where g and $h : A \oplus X \rightarrow B \oplus U$ are continuous linear maps, then there are no proper continuous Jorden (g,h)-derivation from $A \oplus X$ into $B \oplus U$.

Proof. The proof is similar to that of [15, Theorem 6.2].

Actually, Proposition 2.1.1 and 2.1.2 lead to the following result.

Proposition 2.1.3

Let $A \oplus X$ be unital symmetric amenability module extension Banach algebra, and let $B \oplus U$ be unital Banach $A \oplus X$ -bimodule. Suppose D, g and $h : A \oplus X \rightarrow B \oplus U$ are continuous linear maps. If D is a (g,h)-derivation at the unit element, then D is a (g,h)-derivation.

The local triple derivations, triple derivations and derivations on C^*-algebras have common property that they transfer the unit element from domain C^*-algebras into a symmetric element see ([6, Lemma 2.2], [12, proof of Lemma 1], [13, Lemma 3.4] and [16, Lemma 2.1]). This property holds to linear maps on $*$-module extension Banach algebras which are (g,h)-derivations at $(1,0)$ as well.

Lemma 2.1.4

Let $D : *\text{-}A \oplus X \rightarrow *\text{-}B \oplus U$ be (g,t,h)-ternary derivation at $(1,0)$ of $*\text{-}A \oplus X$, then the following arguments hold:

(i) $D(1,0)^* = -D(1,0)$;

(ii) The identity $D(\text{ş},\text{ŋ}) = \frac{1}{2}(D(\text{ş},\text{ŋ})\, h(\text{ş},\text{ŋ}) + g(\text{ş},\text{ŋ})\, D(\text{ş},\text{ŋ}) - g(\text{ş},\text{ŋ})\, D(1,0)\, h(\text{ş},\text{ŋ}) + D(\text{ş},\text{ŋ})\, g(\text{ş},\text{ŋ}) + h(\text{ş},\text{ŋ})\, D(\text{ş},\text{ŋ}) - h(\text{ş},\text{ŋ})\, D(1,0)\, g(\text{ş},\text{ŋ}))$, holds for all projection $(\text{ş},\text{ŋ})$ in $*\text{-}A \oplus X$.

Proof. (i) Since $(1,0) = \{(1,0),(1,0),(1,0)\}$, by the hypothesis, we get $D(1,0) = D\{(1,0),(1,0),(1,0)\} = \{D(1,0), t(1,0), h(1,0)\} + \{g(1,0), D(1,0), h(1,0)\} + \{g(1,0), t(1,0), D(1,0)\} = 2\, D(1,0) + D(1,0)^*$. This implies that $D(1,0)^* = -D(1,0)$.

(ii) Suppose a projection $(\text{ş},\text{ŋ}) \in *\text{-}A \oplus X$, and let the identity $(1,0) = \{((1,0) - 2(\text{ş},\text{ŋ})), (1,0), ((1,0) - 2(\text{ş},\text{ŋ}))\}$. From assumptions, we have that $D(1,0) = D\{((1,0) - 2(\text{ş},\text{ŋ})),\ (1,0),\ ((1,0) - 2(\text{ş},\text{ŋ}))\} = \{D((1,0) - 2(\text{ş},\text{ŋ})),\ t(1,0),\ h((1,0) - 2(\text{ş},\text{ŋ}))\} + \{g((1,0) - 2(\text{ş},\text{ŋ})), D(1,0),\ h((1,0) - 2(\text{ş},\text{ŋ}))\} + \{g((1,0) - 2(\text{ş},\text{ŋ})),\ t(1,0), D((1,0) - 2(\text{ş},\text{ŋ}))\} = D(1,0) - 4\, D(\text{ş},\text{ŋ}) + 2\, D(\text{ş},\text{ŋ})\, h(\text{ş},\text{ŋ}) + 2\, h(\text{ş},\text{ŋ})\, D(\text{ş},\text{ŋ}) + 2\, g(\text{ş},\text{ŋ})\, D(\text{ş},\text{ŋ}) + 2\, D(\text{ş},\text{ŋ})\, g(\text{ş},\text{ŋ}) - 2\, g(\text{ş},\text{ŋ})\, D(1,0)\, h(\text{ş},\text{ŋ}) - 2\, h(\text{ş},\text{ŋ})\, D(1,0)\, g(\text{ş},\text{ŋ})$. Hence,

$D(\text{ş},\text{ŋ}) = \frac{1}{2}(D(\text{ş},\text{ŋ})\, h(\text{ş},\text{ŋ}) + g(\text{ş},\text{ŋ})\, D(\text{ş},\text{ŋ}) - g(\text{ş},\text{ŋ})\, D(1,0)\, h(\text{ş},\text{ŋ}) + D(\text{ş},\text{ŋ})\, g(\text{ş},\text{ŋ}) + h(\text{ş},\text{ŋ})\, D(\text{ş},\text{ŋ}) - h(\text{ş},\text{ŋ})\, D(1,0)\, g(\text{ş},\text{ŋ}))$. ∎

There exist $*$-module extension Banach algebras containing just zero projection. Therefore, we necessary to transact with unitaries.

Theorem 2.1.5

Let a bounded linear map $D: *\text{-}A \oplus X \to *\text{-}B \oplus U$ be a (g,t,h)-ternary derivation at the unit of $* - A \oplus X$, then D is (g,h)-generalized derivation.

Proof. Let us fix $(ş, ŋ) \in (* - A \oplus X)_{sa}$. For all $\lambda \in R$, $(e^{i\lambda(ş,ŋ)}, 0)$ is a unitary in $*\text{-}A \oplus X$ and $(1,0) = \{ (e^{i\lambda(ş,ŋ)}, 0), (1,0), (e^{-i\lambda(ş,ŋ)}, 0)\}$. Therefore,

$D(1,0) = \{D(e^{i\lambda(ş,ŋ)}, 0), t(1,0), h(e^{-i\lambda(ş,ŋ)}, 0)\} + \{g(e^{i\lambda(ş,ŋ)}, 0), D(1,0), h(e^{-i\lambda(ş,ŋ)}, 0)\} + \{g(e^{i\lambda(ş,ŋ)}, 0), t(1,0), D(e^{-i\lambda(ş,ŋ)}, 0)\}$. By taking the first derivative at λ, we have that

$$\begin{aligned}(0,0) &= \left\{D\left((ş,ŋ)(e^{i\lambda(ş,ŋ)},0)\right), t(1,0), h(e^{-i\lambda(ş,ŋ)},0)\right\} \\ &\quad -\left\{D(e^{i\lambda(ş,ŋ)},0),\ t(1,0),\ h\left((ş,ŋ)(e^{i\lambda(ş,ŋ)},0)\right)\right\} \\ &\quad +\left\{g\left((ş,ŋ)(e^{i\lambda(ş,ŋ)},0)\right), D(1,0), h(e^{-i\lambda(ş,ŋ)},0)\right\} \\ &\quad -\left\{g(e^{i\lambda(ş,ŋ)},0), D(1,0),\ h\left((ş,ŋ)\ (e^{i\lambda(ş,ŋ)},0)\right)\right\} \\ &\quad +\left\{g\left((ş,ŋ)(e^{i\lambda(ş,ŋ)},0)\right), t(1,0), D(e^{-i\lambda(ş,ŋ)},0)\right\} \\ &\quad -\left\{g(e^{i\lambda(ş,ŋ)},0),\ t(1,0),\ D\left((ş,ŋ)(e^{i\lambda(ş,ŋ)},0)\right)\right\}.\end{aligned}$$

Let`s take the second derivative at $\lambda = 0$ in the final equality, we get $(0,0) = 2\{D(ş,ŋ)^2,\ t(1,0), h(1,0)\} - 2\{D(ş,ŋ),\ t(1,0), h(ş,ŋ)\} - 2\{g(ş,ŋ),\ D(1,0),\ h(1,0)\} - 2\{g(ş,ŋ), t(1,0), D(ş,ŋ)\}$, or equivalent $2\ D(ş,ŋ)^2 = D(ş,ŋ)\ h(ş,ŋ) + h(ş,ŋ)\ D(ş,ŋ) + g(ş,ŋ)\ D(ş,ŋ) + D(ş,ŋ)\ g(ş,ŋ) + g(ş,ŋ)\ D(1,0)^*\ h(ş,ŋ) + h(ş,ŋ)\ D(1,0)^*\ g(ş,ŋ)$. From Lemma 2.1.4 (i), we have that $D(ş,ŋ)^2 = \frac{1}{2}\big(D(ş,ŋ)\ h(ş,ŋ) + g(ş,ŋ)\ D(ş,ŋ) - g(ş,ŋ)\ D(1,0)\ h(ş,ŋ) + D(ş,ŋ)\ g(ş,ŋ) + h(ş,ŋ)\ D(ş,ŋ) - h(ş,ŋ)\ D(1,0)\ g(ş,ŋ)\big)$,

(1)

for all $(ş,ŋ)$ in $(* - A \oplus X)_{sa}$. Now, let us choose $(ş,ŋ), (ŗ,ɱ), (ç,ʐ)$ in $(* - A \oplus X)_{sa}$ such that $(ş,ŋ)(ŗ,ɱ) = (0,0) = (ŗ,ɱ)(ç,ʐ)$ and $g(ş,ŋ)h(ŗ,ɱ) = (0,0) = g(ŗ,ɱ)\ h(ç,ʐ)$. If we take $(ŗ,ɱ) = (ŗ,ɱ)^+ - (ŗ,ɱ)^-$, with $(ŗ,ɱ)^+\ (ŗ,ɱ)^- = (0,0)$ and $(ŗ,ɱ)^\alpha \geq (0,0)$ for all $\alpha \in \{\mp\}$, find $(ƒ,ķ)^\alpha \geq (0,0)$ in $*\text{-}A \oplus X$ such that $((ƒ,ķ)^\alpha)^2 = (ŗ,ɱ)^\alpha$ ($\alpha = \mp$). It is easily to check that $(ş,ŋ)\ (ƒ,ķ)^\alpha = (0,0) = (ƒ,ķ)^\alpha\ (ç,ʐ)$ and $g(ş,ŋ)\ h((ƒ,ķ)^\alpha) = (0,0) = g((ƒ,ķ)^\alpha)\ h(ç,ʐ)$ for $\alpha = \mp$, by applying (1) we have that $g(ş,ŋ)\ D(ŗ,ɱ)\ h(ç,ʐ) = g(ş,ŋ)\ D((ŗ,ɱ)^+)\ h(ç,ʐ) - g(ş,ŋ)\ D((ŗ,ɱ)^-)\ h(ç,ʐ) = \frac{1}{2}(g(ş,ŋ)(\ D((ƒ,ķ)^+)\ h((ƒ,ķ)^+) + g((ƒ,ķ)^+)\ \ D((ƒ,ķ)^+)\ - \ g((ƒ,ķ)^+)\ \ D((ƒ,ķ)^+)\ \ h((ƒ,ķ)^+)\ + D((ƒ,ķ)^+)\ \ g((ƒ,ķ)^+) + h((ƒ,ķ)^+)\ \ D((ƒ,ķ)^+)\ - \ h((ƒ,ķ)^+)\ \ D((ƒ,ķ)^+)\ \ g((ƒ,ķ)^+)\ )\ \ h(ç,ʐ)$ $\frac{1}{2}(g(ş,ŋ)(D((ƒ,ķ)^-)h((ƒ,ķ)^-) + g((ƒ,ķ)^-)\ D((ƒ,ķ)^-) - (g(ƒ,ķ)^-\ )\ D((ƒ,ķ)^-)\ h((ƒ,ķ)^-)\ + D((ƒ,ķ)^-)\ \ g((ƒ,ķ)^-)\ + h((ƒ,ķ)^-\ )\ D((ƒ,ķ)^-) - h((ƒ,ķ)^-)\ D((ƒ,ķ)^-)\ \ g((ƒ,ķ)^-)\ )\ h(ç,ʐ)) = (0,0)$. Hence, $g(ş,ŋ)\ D(ŗ,ɱ)\ h(ç,ʐ) = (0,0)$.

We deduce from Proposition 2.1 that D is a (g,h)-generalized derivation.

There exist (g,h)-generalized derivations on $*$-module extension Banach algebra which are not (g,t,h)-ternary derivations at $(1,0)$, see the following example.

Example 2.1.6

Let $g,t,h : * \text{ - } A \oplus X \to * \text{ - } A \oplus X$ be continuous homomorphisms. Suppose element $(ş, ŋ)$ is a non-zero symmetric in $*$-$A \oplus X$ and let D be a continuous linear map on $*$-$A \oplus X$ defined by $D(ŗ, ɱ) = (ş, ŋ)\ h(ŗ, ɱ)$, for all $(ŗ, ɱ)$ in $*$-$A \oplus X$. Then

$$D\big((ŗ, ɱ)\ (ç, ʐ)\big) = (ş, ŋ)\ h\big((ŗ, ɱ)(ç, ʐ)\big). \quad (2)$$

$$D(ŗ, ɱ)h(ç, ʐ) + g(ŗ, ɱ)\ D(ç, ʐ) - g(ŗ, ɱ)\ D(1,0)\ h(ç, ʐ) = (ş, ŋ)\ h(ŗ, ɱ)\ h(ç, ʐ) + g(ŗ, ɱ)\ (ş, ŋ)\ h(ç, ʐ) - g(ŗ, ɱ)\ (ş, ŋ)\ h(1,0)\ h(ç, ʐ) = (ş, ŋ)\ h((ŗ, ɱ)(ç, ʐ)), \quad (3)$$

for all $(ŗ, ɱ), (ç, ʐ)$ in $* - A \oplus X$. From equations (2) and (3), we have that, D is a (g,h)-generalized derivation. Moreover, $D(1,0) = (ş, ŋ) \in (* - A \oplus X\ )_{sa} \setminus \{(0,0)\}$ with Lemma 2.1.4, we have that D is not (g,t,h)-ternary derivation at $(1,0)$.

Whenever $D(1,0) = (0,0)$, a proper modification to the arguments stated in Theorem 2.1.5 provides extra information.

Proposition 2.1.7

Let a bounded linear map $D : *\text{-}A \oplus X \to *\text{-}A \oplus X$ be a (g,t,h)-ternary derivation at the unit of $*$-$A \oplus X$ with $D(1,0) = (0,0)$, then D is a $*$-(g,h)-derivation and a (g,t,h)-ternary derivation.

Proof. In the same way as in the proof of Theorem 2.1.5, let us pick $(ş, ŋ) \in (* - A \oplus X)_{sa}$. For all $\lambda \in R$, $(e^{i\lambda(ş,ŋ)}, 0)$ is a unitary in $*$ - $A \oplus X$ and $(1,0) = \{\ (e^{i\lambda(ş,ŋ)}, 0), (e^{i\lambda(ş,ŋ)}, 0), (1,0)\}$. Thus,

$$\begin{aligned}(0,0) &= D(1,0)\\ &= \{\ D(e^{i\lambda(ş,ŋ)}, 0), t(e^{i\lambda(ş,ŋ)}, 0), h(1,0)\} + \{g(e^{i\lambda(ş,ŋ)}, 0), D(e^{i\lambda(ş,ŋ)}, 0), h(1,0)\} + \{g(e^{i\lambda(ş,ŋ)}, 0), t(e^{i\lambda(ş,ŋ)}, 0), D(1,0)\},\\ &= \tfrac{1}{2}(D(e^{i\lambda(ş,ŋ)}, 0)\ t(e^{i\lambda(ş,ŋ)}, 0)^* + t(e^{i\lambda(ş,ŋ)}, 0)^*\ D(e^{i\lambda(ş,ŋ)}, 0),\\ &\quad + \tfrac{1}{2}(g(e^{i\lambda(ş,ŋ)}, 0)D(e^{i\lambda(ş,ŋ)}, 0)^* + D(e^{i\lambda(ş,ŋ)}, 0)^* g(e^{i\lambda(ş,ŋ)}, 0)).\end{aligned}$$

That is, $(0,0) = D(e^{i\lambda(ş,ŋ)}, 0) \circ t(e^{-i\lambda(ş,ŋ)}, 0) + g(e^{i\lambda(ş,ŋ)}, 0) \circ D(e^{i\lambda(ş,ŋ)}, 0)^*$. By taking a derivative at $\lambda = 0$, $(0,0) = D(ş, ŋ) - D(1,0) \circ t(ş, ŋ) + g(ş, ŋ) \circ D(1,0)^* - D(ş, ŋ)^*$. That is implies, $D(ş, ŋ) = D(ş, ŋ)^*$ for every $(ş, ŋ) \in (* - A \oplus X)_{sa}$. Therefore, D is a symmetric map. We have from Theorem 2.1.5, that D is a (g,h)-generalized derivation. Since D is symmetric map and $D(1,0) = (0,0)$, then D is a $*$-(g,h)-derivation and also a (g,t,h)-ternary (triple) derivation.

Corollary 2.1.8

Let $D : *\text{-}A \oplus X \to *\text{-}A \oplus X$ be a bounded linear map on unital $*$-$A \oplus X$ that is (g,t,h)-ternary derivation at the unit of $*$-$A \oplus X$, then D is a (g,t,h)-ternary derivation.

Proof. The fact that the mapping $d\big(D(1,0), (1,0)\big) : * \text{ - } A \oplus X \to * \text{ - } A \oplus X$ defined by $d\big(D(1,0), (1,0)\big)\ (ş, ŋ) = \{D(1,0), t(1,0), h(ş, ŋ)\} -$

$\{g(1,0), D(1,0), h(ş,ŋ)\}$, and by Lemma 2.1.4, $D(1,0)^* = -D(1,0)$. We have d is a (g,t,h)-ternary derivation see [12, Proof of Lemma 1]. The real linear combination of linear maps that are (g,t,h)-ternary derivations at (1,0) is also a (g,t,h)-ternary derivation at (1,0). Thus, the following mapping $\widetilde{D} = D - \frac{1}{2} d\big(D(1,0),(1,0)\big)$ is a (g,t,h)-ternary derivation at (1,0) and $\widetilde{D}(1,0) = (0,0)$. By Proposition 2.1.7, we get $\widetilde{D}$ is a $*$-(g,h)-derivation and a (g,t,h)-ternary derivation as well. Hence, the map $D = \widetilde{D} + \frac{1}{2} d\big(D(1,0),(1,0)\big)$ is a (g,t,h)-ternary derivation. ■

Corollary 2.1.9 [6, Theorem 2.3]

Let D be a bounded linear map from a C^*-algebra A into A-bimodule B which is a ternary derivation at the unit of A, then D is a generalized derivation.

Proof. By Theorem 2.1.5, taking g, t and h to be the identity maps and $X = U = 0$.

Corollary 2.1.10 [6, Proposition 2.4]

If a bounded linear map D defined on a C^*-algebra A is a ternary derivation at the unit element of A with $D(1) = 0$, then D is a $*$-derivation and a ternary derivation.

Proof. By proposition 2.1.7, taking g, t and h to be the identity maps and $X = U = 0$.

Corollary 2.1.11. [6, Corollary 2.5]

Let a bounded linear map D defined on unital C^*-algebra A be a ternary derivation at the unit element of A, then D is a ternary derivation.

Proof. By Corollary 2.1.8, taking g, t and h to be the identity maps and $X = U = 0$.

2.2 (g,t,h)-Ternary Derivation of a $*$-Module Extension Banach Algebra at Zero

We will explore the fundamental properties of (g,h)-derivations at zero on a $*$-module extension Banach algebra in this subsection.

Lemma 2.2.1

Let $*$-$A \oplus X$ be a $*$-submodule extension Banach algebra of a $*$-module extension Banach algebra $*$-$B \oplus U$, and let a linear map $D : *\text{-}A \oplus X \to *\text{-}B \oplus U$ be (g,h)-derivation at zero, where $g,\ h : *\text{-}A \oplus X \to *\text{-}B \oplus U$ are homeomorphisms, then $g(ş,ŋ)\, D(ŗ,ɱ)\ h(ç,ʐ) = (0,0), \forall\, (ş,ŋ),(ŗ,ɱ),(ç,ʐ) \in *\text{-}A \oplus X$, such that $(ş,ŋ)\,(ŗ,ɱ) = (ŗ,ɱ)\ (ç,ʐ) = (0,0)$.

Proof. Let us take $(ş,ŋ)$, $(ŗ,ɱ)$, $(ç,ʐ)$ in $*$-$A \oplus X$ such that $(ş,ŋ)\,(ŗ,ɱ) = (ŗ,ɱ)\ (ç,ʐ) = (0,0)$. Since D is (g,h)-derivation at zero, we get

$g(ş,ŋ)\, D(ŗ,ɱ)\ h(ç,ʐ) = \big(D\big((ş,ŋ)(ŗ,ɱ)\big) - D(ş,ŋ)\ h(ŗ,ɱ)\big) h(ç,ʐ) = (0,0)$. ■

Let's note that under the supposition of the aforementioned lemma, we are unable to apply Proposition 2.1 because D is not considered to be bounded.

Recall that $(ş,ŋ)$ and $(ŗ,ɱ)$ in $*$-$A \oplus X$ are orthogonal (denoted by $(ş,ŋ) \perp (ŗ,ɱ)$) if and only if $(ş,ŋ)\,(ŗ,ɱ)^* = (ŗ,ɱ)^*\,(ş,ŋ) = (0,0)$.

Lemma 2.2.2

Let $*$-$A \oplus X$ be a $*$-submodule extension Banach algebra of a $*$-module extension Banach algebra $*$-$B \oplus U$, and let a linear map $D : *\text{-}A \oplus X \to *\text{-}B \oplus U$ be (g,t,h)-ternary derivation at zero, where $g,\ t,\ h : *\text{-}A \oplus X \to *\text{-}B \oplus U$ are homomorphisms, then $\{g(ş,ŋ), D(ŗ,ɱ), h(ç,ʐ)\} = (0,0)$, $\forall\, (ş,ŋ)$,

$(ŗ, ɱ)$, $(ç, ʐ) \in *\text{-}A \oplus X$ such that $(ş, ŋ) \perp (ŗ, ɱ) \perp (ç, ʐ)$ and $g(ş, ŋ) \perp t(ŗ, ɱ) \perp h(ç, ʐ)$.

Proof. Assume that $(ş, ŋ)$, $(ŗ, ɱ)$, $(ç, ʐ)$ in $*\text{-}A \oplus X$, satisfying $(ş, ŋ) \perp (ŗ, ɱ) \perp (ç, ʐ)$ and $g(ş, ŋ) \perp t(ŗ, ɱ) \perp h(ç, ʐ)$. Since $\{(ş, ŋ), (ŗ, ɱ), (ç, ʐ)\} = (0,0)$, this implies that $(0,0) = \{D(ş, ŋ),\ t(ŗ, ɱ),\ h(ç, ʐ)\} + \{g(ş, ŋ),\ D(ŗ, ɱ),\ h(ç, ʐ)\} + \{g(ş, ŋ),\ t(ŗ, ɱ),\ D(ç, ʐ)\} = \{g(ş, ŋ), D(ŗ, ɱ), h(ç, ʐ)\}$. Since, $\{D(ş, ŋ),\ t(ŗ, ɱ),\ h(ç, ʐ)\} = \{g(ş, ŋ), t(ŗ, ɱ), D(ç, ʐ)\} = (0,0)$. Therefore, $\{g(ş, ŋ), D(ŗ, ɱ), h(ç, ʐ)\} = (0,0)$. This proved the statement. ■

Theorem 2.2.3

Let $*\text{-}A \oplus X$ be a $*$-submodule extension Banach algebra of a unital $*$-module extension Banach algebra $*\text{-}B \oplus U$, and let $D : *\text{-}A \oplus X \to *\text{-}B \oplus U$ be a continuous linear map, where g, t and $h : *\text{-}A \oplus X \to *\text{-}B \oplus U$ are continuous homomorphisms. If D is (g, h)-derivation at zero or (g, t, h)-ternary derivation at zero, then D is (g, h)-generalized derivation.

Proof. Suppose D is (g, t, h)-ternary derivation at zero, from Lemma 2.2.2, we have that for all $(ş, ŋ)$, $(ŗ, ɱ)$, $(ç, ʐ) \in (* - A \oplus X)_{sa}$ such that $(ş, ŋ)\,(ŗ, ɱ) = (ŗ, ɱ)\,(ç, ʐ) = (0,0)$ with $g(ş, ŋ) \perp t(ŗ, ɱ) \perp h(ç, ʐ)$ given $(0,0) = 2\{\ g(ş, ŋ),\ \ D(ŗ, ɱ),\ h(ç, ʐ)\ \} = g(ş, ŋ)\ \ D(ŗ, ɱ)^*\ \ h(ç, ʐ) + h(ç, ʐ)\ \ D(ŗ, ɱ)^*\ \ g(ş, ŋ)$, or equivalently, $(0,0) = g(ş, ŋ)\, D(ŗ, ɱ)\, h(ç, ʐ) + h(ç, ʐ)\, D(ŗ, ɱ)\, g(ş, ŋ)$. When we assume that D is (g, h)-derivation at $(0,0)$. From Lemma 2.2.1, we get $g(ş, ŋ)\, D(ŗ, ɱ)\, h(ç, ʐ) = (0,0)$. It follows from Proposition 2.1 (4) ⇔ (1), assures that D is (g, h)-generalized derivation. ■

Now, we can say that Theorem 2.2.3, extends [6, Theorem 2.9] and [14, Theorem 4] by using
of unital $*$-module extension Banach algebras.

Corollary 2.2.4 [6, Theorem 2.9]

Suppose A is C^*-subalgebra of unital C^*-algebra B, and let a continuous linear map $D : A \to B$ be derivation at zero or ternary derivation at zero, then D is generalized derivation.

Proof. Applying Theorem 2.2.3, taking g, t and h to be the identity maps and $X = U = 0$.

Reference

[1] C.A. Akemann, "The General Stone–Weierstrass Problem, " Journal Functional Analysis, vol. 4, pp. 277–294, 1969.

[2] J. Alaminos, M. Bresar, J. Extremera and A. Villena, "Maps Preserving Zero Products, " Studia Mathematica, vol. 193, no. 2, pp. 131–159, 2009.

[3] S. Ayupov, K. Kudaybergenov and A.M. Peralta, "A Survey on Local and 2-Local Derivations on C∗- and Von Neuman Algebras," in: Contemporary Mathematica, vol. 672, pp. 73–126, 2016.

[4] D.P. Blecher and M. Neal, "Open Partial Isometries and Positivity in Operator Spaces," Studia Mathematica, vol. 182, no. 3, pp. 227–262, 2007.

[5] M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, "Local triple derivations on C∗-algebras", Communications in Algebra, vol. 42, no. 3, pp. 1276–1286, 2014.

[6] A. B. A. Essaleh and A. M. Peralta, "Linear Maps on C*-Algebras Which are Derivations or Triple Derivations at a Point, " Linear Algebra and its Applications, vol. 538, pp. 1-21, 2018.

[7] A.B.A. Essaleh, A.M. Peralta and M.I. Ramírez, "Weak-local derivations and homomorphisms on C∗-algebras", Linear Multilinear Algebra, vol. 64, no. 2, pp. 169–186, 2016.

[8] F.J. Fernández Polo and A.M. Peralta, "Non-Commutative Generalisations of Urysohn’s Lemma and Hereditary Inner Ideals," Journal of Functional Analysis, vol. 259, pp. 343–358, 2010.

[9] L.A. Harris, " Bounded Symmetric Homogeneous Domains in Infinite Dimensional Spaces," in: Proceedings on Infinite Dimensional Holomorphic, International conference University Kentucky, Lexington, Ky., 1973, in: Lecture Notes in Mathematica, vol. 364, pp. 13–40, 1974.

[10] L.A. Harris, " A Generalization of C∗-Algebras, " Proceedings London Mathematica, vol. 3, no. 42(2), pp. 331–361, 1981.

[11] M. Hestenes, "A Ternary Algebra with Applications to Matrices and Linear Transformations, " Archive for Rational Mechanics and Analysis, vol. 11, pp. 138–194, 1962.

[12] T. Ho, J. Martinez-Moreno, A.M. Peralta and B. Russo, " Derivations on Real and Complex JB∗-Triples", Journal of the London Mathematical, vol. 2, no. 65(1), pp. 85–102, 2002.

[13] T. Ho, A.M. Peralta and B. Russo, "Ternary Weakly Amenable C∗-Algebras and JB∗-Triples, " Quarterly Journal of Mathematical, vol. 64, no. 4, pp. 1109–1139, 2013.

[14] W. Jing, S.J. Lu and P.T. Li, "Characterization of Derivations on Some Operator Algebras, " Bulletin of the Australian Mathematical, vol. 66, pp. 227–232, 2002.

[15] B.E. Johnson, "Symmetric Amenability and the Nonexistence of Lie and Jordan Derivations, " Mathematical Proceedings of the Cambridge Philosophical Society, vol. 120, no. 3, pp. 455–473, 1996.

[16] K. Kudaybergenov, T. Oikhberg, A.M. Peralta and B. Russo, "2-Local Triple Derivations on Von Neumann Algebras", Illinois Journal of Mathematics, vol. 58, no. 4, pp. 1055–1069, 2014.

[17] J. Li and Z. Pan, "Annihilator-Preserving Maps, Multipliers, and Derivations, " Linear Algebra and its Applications, vol. 423, pp. 5–13, 2010.

[18] F. Lu, "Characterizations of Derivations and Jordan Derivations on Banach Algebras, " Linear Algebra and its Applications, vol. 430, no. (8–9), pp. 2233–2239, 2009.

[19] A. R. Medghalchi and H. Pourmahmood-Aghababa, "The First Cohomology Group of Module Extension Banach Algebras, " Rocky Mountain Journal of Mathematical, vol. 41, pp. 1639–1651, 2011.

[20] M. Neal and B. Russo, "Operator Space Characterizations of C∗-Algebras and Ternary Rings, " Pacific Journal of Mathematics, vol. 209, no. 2, pp. 339–364, 2003.

[21] C. Parky and D. Yun shinz, "Generalized (θ, ø)-Derivation on Banach Algebras", Korean Journal of Mathematics, vol. 22, no. 1, pp. 139-150, 2014.

[22] G.K. Pedersen, "C∗-Algebras and Their Automorphism Groups, " London Mathematical Society Monographs, vol. 14, Academic Press, London, 1979.

[23] S. Sakai, "C∗-Algebras and W∗-Algebras, " Springer-Verlag, Berlin, 1971.

[24] M. Takesaki, "Theory of Operator Algebras I, " Springer-Verlag, Berlin, 1979.

[25] H. Zettl, "A Characterization of Ternary Rings of Operators, " Advances in Mathematics, vol. 48, pp. 117–143, 1983.

[26] Y. Zhang, "Weak Amenability of Module Extensions of Banach Algebras, " Transactions of the American Mathematical Society, vol. 354, pp. 4131–4151, 2002.