Co-Secure Regular Set Domination in Graphs

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Abstract

Here, we initiate the new domination parameter called Co-secure regular set dominating set and is defined as: A Co-secure dominating set D of G is said to be a Co-secure regular set dominating set of G if for every set $T \subseteq V - D$ there exists a non-void subset $S \subseteq D$ then the subgraph $< T \cup S >$ induced by $T \cup S$ is regular and is denoted as CSRSD-set. The $\gamma_{cs}^{rs}(G)$, the Co-secure regular set domination number of G is the cardinality of least CSRSD-set in G. Also, we obtained the $\gamma_{cs}^{rs}(G)$ of various standard graphs and sharp bounds. Also characterize the $\gamma_{cs}^{rs}(T) = m - 2$ for all trees.
Keywords: *Co-Secure dominating set, Corona Product, Regular Set Domination.*

I Introduction

Consider $G = (V, E)$, the simple, undirected finite graph throughout this paper. For a vertex $v \in V$, the $deg(v)$ is the number of edges connected to a vertex v.

The P_m , path having m vertices. A graph of order m is a complete graph, K_m in which every vertex is of degree $m - 1$. The $K_{1,m}$, star graph with $m + 1$ vertices. A complete bipartite graph, $K_{m,n}$ with M and N as a partite set with cardinality m and n respectively. The F_m , a friendship graph is obtained by joining the m copies of K_3 to a common vertex with degree 2p and all other vertex having degree 2. The Corona product of two distinct graph G and H, give rise to a new graph by considering G's one copy along with $|V(G)|$ times of H, then by connecting the i^{th} vertex of G to all the vertex of i^{th} copy of H.

The set $D \ in\ V(G)$ is a dominating set of G (or D-Set) if every u in $V - D$ is dominated by some vertex in D. The $\gamma(G)$, domination number is the size of a least D- set of G. A subset $D \subseteq V(G)$ is a regular set dominating set if for each set $T \subseteq V - D$ there exist a set S in D such that the subgraph induced by <T∪S> is regular and for $|T| = 1$, the $< T \cup S >$ is 1- regular and is abbreviated as RSD-set. The regular set domination number, $\gamma_{rs}(G)$ is the cardinality of least RSD-set of G[4]. V. R. Kulli et al.,[4] initiated the RSD-set and they determined the $\gamma_{rs}(G)$ of various other standard graphs. They obtained its sharp bounds. A dominating set $D \subseteq V(G)$ is a Co-secure dominating set of G if for each $u' \in D$ there exist $v' \in V \setminus D$ with $u'v'$ is an edge and $(D \setminus \{u'\}) \cup \{v'\}$ is a D-set[2] and is denoted as CSD-set. The size of a least CSD-set in G is the Co-Secure domination number, $\gamma_{cs}(G)$ of G. In many practical situations we have to substitute the guard in the dominating set by another guard in the museum and this made S. Arumugam and et al.,[2] to introduce the concept Co-secure domination set and in that they determined the $\gamma_{cs}(G)$ of various standard graph and for this parameter they obtain the sharp bounds. Aleena Joseph and Sangeetha investigated the co-secure domination number of Friendship graph, Jahangir graph and Helm graph and also obtained the bounds[1]. Sunilkumar M.Hosamani[7] introduced the

Degree Equitable Regular Set Domination parameter and they find the $\gamma_{rs}^{e}(G)$ for some standard graphs and obtain sharp bounds and also they determine the solution to the open problem proposed by Sampathkmar and Pushpalatha[5]. Seema Mehra and Deepak[6] introduced the Secure Regular Set dominating set and find the Secure regular set domination number of some standard graphs and its bounds[6]. This motivated us to initiate the "Co-Secure Regular Set Domination Set". It was introduced fully on theoretical aspects but it has both the applications of two domination parameter.

II Main Results

Here, we initiate and investigate the $\gamma_{cs}^{rs}(G)$ of some standard graphs. However, this number,$\gamma_{cs}^{rs}(G)$ can never be equal to an order of G.

Definition 2.1

A CSD-set 'D' of G is said to be a Co-secure regular set dominating set of G if for every $T \subseteq V - D$ there exists a non-void subset $S \subseteq D$ such that the subgraph $< T \cup S >$ induced by $T \cup S$ is regular and is abbreviated as (CSRSD-set). The cardinality of least CSRSD-set is a co-secure regular set domination number, $\gamma_{cs}^{rs}(G)$ of G.

It is clear that the dominating set D is CSRSD-set if D behave as both CSD-set and RSD-set. It is observed that there will be no CSRSD-set for a graph G having an isolated vertex. Here, we consider only the connected non-trivial graph G. For undefined terminology refer in [3].

Example 2.2

For the graph G_1 given in figure 1, we have $\{v_4, v_7\}$ as a dominating set and is the only $\gamma -$set of G_1. Therefore $\gamma(G_1) = 2$. Also $\{v_4, v_5, v_7\}$ is a $\gamma_{cs} -$ set and $\gamma_{cs}(G_1) = 3$. The set $\{v_2, v_3, v_5, v_6, v_8, v_9\}$ is a $\gamma_{cs}^{rs} -$set of G_1. Hence $\gamma_{cs}^{rs}(G_1) = 6$.In figure 2, the graph G_2 have $\gamma(G_2) = 2, \gamma_{cs}(G_2) = 5$ and $\gamma_{cs}^{rs}(G_2) = 5$. In fact, $D_1 = \{v_2, v_5\}$ is a $\gamma -$set of G_2, $D_2 = \{v_1, v_2, v_3, v_6, v_7\}$ as a $\gamma_{cs} -$ set of G_2 and $D_3 = \{v_1, v_2, v_3, v_6, v_7\}$ as a $\gamma_{cs}^{rs} -$set of G_2.

Figure 1. A graph G_1 with $\gamma = 2, \gamma_{cs} = 3, \gamma_{cs}^{rs} = 6$.

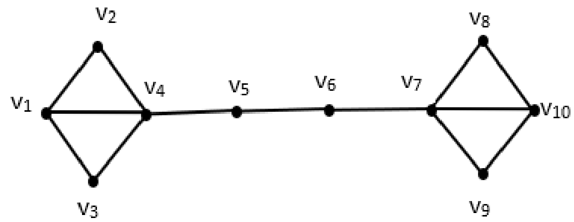
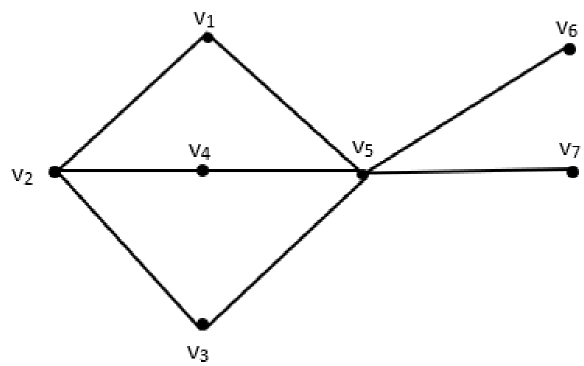


Figure 2. A graph G_2 with $\gamma = 2, \gamma_{cs} = 5, \gamma_{cs}^{rs} = 5$.



Since every CSRSD-set is always a dominating set and we have,

Observation 2.3

For any non-isolated graph G, the $\gamma(G) \leq \gamma_{cs}(G) \leq \gamma_{cs}^{rs}(G)$.

Theorem 2.4

If G, a graph with non-isolated vertex and order m, then $1 \leq \gamma_{cs}^{rs}(G) \leq m - 1$. Further, $\gamma_{cs}^{rs}(G) = 1$ if and only if $G = K_m, m \geq 2$ and $\gamma_{cs}^{rs}(G) = m - 1$ if and only if $G = K_{1,m-1}$.

Proof

The bounds of $\gamma_{cs}^{rs}(G)$ are trivial. Assume that D is a CSRSD-set of G with $\gamma_{cs}^{rs}(G) = 1 = |D|$, so we take $D = \{v_1\}$. Suppose there exists any two vertices $v_i, v_j \in V - D$ are not adjacent to each other, means that the subgraph induced by $D \cup \{v_j, v_i\}$ is not regular, a contradiction. So, every vertex in $V - D$ are adjacent to each other and also adjacent to the vertex v_1 in D. Since $D = \{v_1\}$ is also a CSD-set and if we restore v_1 by any other vertex $\{v_i\}$ in $V - D$ for $i \geq 2$ is also D-set. Therefore, each and every vertex in the graph G is of degree $m - 1$. Hence the graph $G = K_m$. Conversely, consider

$G = K_m$, then every $\{v_i\}$ for $i \geq 1$ can be a CSRSD- set and hence $\gamma_{cs}^{rs}(G) = 1$.

If $G = K_{1,m-1}$ then trivially the set of all leaf vertices of G is a smallest CSRSD-set of G and therefore $\gamma_{cs}^{rs}(G) = m - 1$. Conversely, Suppose $\gamma_{cs}^{rs}(G) = m - 1 = |D|$. Consider D as a CSRSD- set with $|D| = m - 1$ and in $V - D$ we have only one element (support vertex), that is $V - D = \{u\}$. Since D is a CSD-set so all the vertices in D are restored by the vertex u in $V - D$. Suppose if there exist a set having two vertices $\{u_1, u_2\} \in D$ and $u_1 u_2 \in E(G)$, then the subgraph formed by $(V - D) \cup \{u_1, u_2\}$ is RSD-set but that D is not a minimum CSD-set because $D \backslash \{u_1\}$ will be a smallest CSD-set, a contradiction. Therefore, all the vertices in D are not adjacent, that means it's an independent set. Hence $G = K_{1,m-1}$.

Observation 2.5

If K_m, a complete graph, then $\gamma_{cs}^{rs}(K_m) = 1$.

Now, we proceed to find the accurate value of $\gamma_{cs}^{rs}(G)$ for some standard graphs.

Observation 2.6

For a path P_m , $\gamma_{cs}^{rs}(P_m) =$

$$\begin{cases} \left\lceil \frac{2m-1}{\Delta(G)-1} \right\rceil, & m = 2,3,5,6,8 \\ \left\lfloor \frac{2m-1}{\Delta(G)-1} \right\rfloor, & m = 9 + i + 3j \text{ , } where\ i = \\ \left\lfloor \frac{2m-1}{\Delta(G)-1} \right\rfloor - 1 & m = 11 + 3j \end{cases}$$

$0,1\ and\ j = \{0\} \cup N$.

And $\gamma_{cs}^{rs}(P_m)$ does not exist for $m = 4\ and\ 7$.

When $m = 4$, for every CSD-set 'D' of P_4, the subgraph $< T \cup S >$ is not regular. i.e., From figure 3 when $D = \{u_2, u_3\}\ or\ D = \{u_1, u_4\}$, then for every set $T = \{u_1, u_4\}$ or $\{u_2, u_3\}$ in $V - D$ we can find a set $S = \{u_2\}\ or\ \{u_2\}\ or\ \{u_2, u_3\}$ in D so that the subgraph $< T \cup S >$ formed is not regular. So $\gamma_{cs}^{rs}(P_4)$ does not exist. Also, for the same reason in P_7,the $\gamma_{cs}^{rs}(P_7)$ does not exist.

Figure 3. A graph P_4 with $\gamma_{cs} = 2$.

Theorem 2.7

If $K_{m,n}$ with $m \leq n$, then $\gamma_{cs}^{rs}(K_{m,n}) = n$.

Proof

Let $M = \{u_1, u_2, \ldots., u_m\}$ and $N = \{v_1, v_2, \ldots . v_n\}$ be the bipartition of $K_{m,n}$. Let us consider D as a CSRSD-set of G. If both $M \cap D$ and $N \cap D$ are non-void set, then we can take $|M \cap D| = m$ and the $|N \cap D| \leq n - m$. For the above cardinality, consider $D = \{u_1, u_2, \ldots \ldots u_m, v_{m+1}, v_{m+2}, \ldots \ldots v_n\}$, and in this case for all the set $T \subseteq V - D$ we can find a set $S \subseteq D$, so that the subgraph formed by $< T \cup S >$ is regular. But no vertex in $V - D$ can replace $v_{m+1}, v_{m+2}, \ldots \ldots v_n$. Hence this D will not be an CSRSD-set. To give the complete proof we need the following two cases.

Case 1

If we consider both the set $M \cap D$ and $N \cap D$ as a non-void set, then we can take $|M \cap D| \leq m - 1$ and $|N \cap D| \leq n - (m - 1)$. For this, each and every set $T \subseteq V - D$ we can form a set $S \subseteq D$ so the newly formed subgraph $< T \cup S >$ is regular. Also, every vertex u in D can be replaced by a v in $V - D$ with $uv \in E(G)$ and $\{D \backslash \{u\}\} \cup \{v\}$ is a D-set. Hence, D is a CSRSD-set with cardinality $|D| = m - 1 + n - (m - 1) = n$.

Case 2

If $M \cap N = \emptyset$ and $N \cap D \neq \emptyset$, so consider $|N \cap D| = n$. Let us consider $D = \{v_1, v_2, \ldots \ldots, v_n\}$ and for all the set $T \subseteq V - D$ we can form a set $S \subseteq D$ so the newly formed subgraph $< T \cup S >$ is regular and this D is a RSD-set. Also, for every u in D can be replaced by a v in $V - D$ with $uv \in E(G)$ and $\{D \backslash \{u\}\} \cup \{v\}$ will also a D-set. Therefore, D is a CSRSD-set with cardinality n.

Hence $\gamma_{cs}^{rs}(K_{m,n}) = n$ for $m \leq n$.

Theorem 2.8

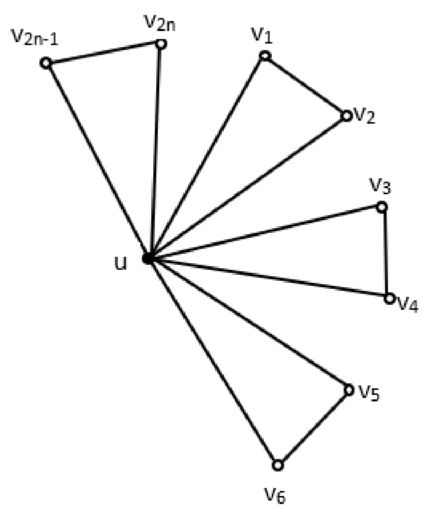
Let $G = F_n$ be a friendship graph, then $\gamma_{cs}^{rs}(F_n) = \frac{m+1}{2} = n + 1$ for $n \geq 2$ and the order of F_n is $m = 2n + 1$. Also, $\gamma_{cs}^{rs}(F_1) = 1$.

Proof

Let F_n, a friendship graph with order $m = 2n + 1$ and each F_n consists of n copies of K_3. Let u be centre vertex and $v_1, v_2, \dots \dots \dots . v_{2n}$ be the vertex labelled in the Figure 4 with $deg(v_i) = 2$, where $i = 1,2, \dots\dots 2n$.
It is trivial that $\gamma_{cs}^{rs}(F_1) = 1$.
Case 1 For $u \notin D$

Let us take of one vertex from each copy of K_3 as the elements of D. Then for each set $T \subseteq V - D$ we have a set S in D so that the induced subgraph $< T \cup S >$ is not a regular. Therefore, D is not a RSD-set. So, there is no CSRSD-set if $u \notin D$.

Figure 4. Friendship graph F_n



Case 2 For $u \in D$

Now D consists of u and one vertex from each copy of K_3. So, every v_{i+1} in $V - D$ is dominated by atmost two vertices u and v_i in D. Thus, any vertex u or v_i in D can be replaced by the vertex v_{i+1} in $V - D$, that is $\{D \backslash \{u\}\} \cup \{v_{i+1}\}$ or $\{D \backslash \{v_i\}\} \cup \{v_{i+1}\}$ is a D-set. Therefore, D is a CSDS. Then every set $T \subseteq V - D$ we have a set $S \subseteq D$ so the newly formed subgraph $< T \cup S >$ is regular. Hence D is a CSRSD-set of $G = F_n$ with cardinality $n + 1$.

Now we have to check that D is the smallest CSRSD-set. To show that assume D as a minimum CSRSD-set having the vertices less than $n + 1$. Consider that D is having a 'n' vertex and also D will be a CSDS as well as RSDS. For this D, all the set T in $V - D$ we have a set S in D so the newly formed subgraph $< T \cup S >$ is a regular graph, a contradiction. So, there is no CSRSD-set with cardinality n. Therefore $\gamma_{cs}^{rs}(F_n) = n + 1$.

Observation 2.9

Let $G = H \circ \overline{K_2}$ and H be any non-trivial graph. If all the edges of H in G are subdivided atmost once, then $\gamma_{cs}^{rs}(G) = 2|V(H)| + |E(H)|$.

III TREES AND BOUNDS FOR CO-SECURE REGULAR SET DOMINATION NUMBERS.

In this section, we determine some basic results for the $\gamma_{cs}^{rs}(T)$ of a tree and also, we determine the upper bounds for the $\gamma_{cs}^{rs}(T)$. In theorem 2.4, we say that if T is a tree of order m and $T \neq K_{1,m-1}$ then $\gamma_{cs}^{rs}(T) \leq m - 2$. In the following theorem we give a characterization of all trees having $\gamma_{cs}^{rs}(T) = m - 2$. If T is a double star, then $\gamma_{cs}^{rs}(T)$ does not exist. Because the induced subgraph formed from the co-secure dominating set is not regular so we subdivided its middle edge and proved the following theorem.

Theorem 3.1

Let T be a tree of order 'm'. Then $\gamma_{cs}^{rs}(T) = m - 2$ if and only if T is obtained by sub-dividing the middle edge of a double star once or twice.

Proof

If T is a tree obtained from a double star by subdividing the edge rs where r and s are the support vertices of degree higher than or equal to 2. Let the new vertex subdividing the edge rs be a 't'. Let L_r and L_s be the set of all pendant vertex adjacent to r and s respectively. Consider $|L_r| \geq 2$ and if $|L_s| \geq 2$, then $L_r \cup L_s \cup \{t\}$ is the unique CSRSD-ser of T. If $|L_s| = 1$ and it have only one pendant vertex u adjacent to s, then $L_r \cup \{u, t\}$ and $L_r \cup \{t, s\}$ are the two CSRSD-set of T. Similarly, if $|L_r| = 1$ and $|L_s| = 1$ then u and v are corresponding pendant vertex of r and s respectively, then $\{u, t, v\}, \{r, t, v\}$ and $\{u, t, s\}$ are the three CSRSD-set of T. Thus, $\gamma_{cs}^{rs}(T) = m - 2$.

Conversely, assume that $\gamma_{cs}^{rs}(T) = m - 2$ and so that D will be a smallest CSRSD-set of T. Also, consider $V - D = \{r, s\}$. Since T is a tree, and by our assumption every vertex in D is adjacent to atmost one vertex in $V - D$. But actually, no vertex in D are adjacent. If we take $deg(r) = deg(s) = 1$, then it represents a path P_4 and from the observation 2.6 it is clear that $\gamma_{cs}^{rs}(T)$ does not exist. Therefore, we can consider rs is not an edge of T, that means $d(r, s) \neq 1$. Suppose $d(r, s) = 2$ and consider $P_1 = \{r, t, s\}$ as a unique r-s path in T. So, T is not a star. Suppose if $deg(r) \geq 1$ and $deg(s) \geq 1$. If $deg(r) = 1$ and $deg(s) = 1$ then $T = P_5$ and in this case $\gamma_{cs}^{rs}(P_5) = 3 = m - 2$. Therefore, T is the tree constructed from a double star by subdividing its middle edge once.

If $d(r, s) = 3$, then $P_2 = \{r, t_1, t_2 s\}$ is a unique r-s path in T. Suppose if $deg(r) \geq 1$ and $deg(s) \geq 1$, then if $deg(r) = deg(s) = 1$ it is P_6. Also, by the observation 2.6 $\gamma_{cs}^{rs}(P_6) = 4 = m - 2$. Hence, we can conclude that T is a tree constructed from a double star by subdividing its middle edge once or twice.

From the theorem 2.4, it follows that for the star $T = K_{1,m-1}$, the $\gamma_{cs}^{rs}(T) = m - 1 = \Delta(T)$. The next theorem gives a characterization for all trees having $\gamma_{cs}^{rs}(T) = \Delta(T) + 1$.

Theorem 3.2

If T, a tree, then $\gamma_{cs}^{rs}(T) = \Delta(T) + 1$ if and only if T is attained by sub-dividing atmost once each edge of $K_{1,p-1}$ $(p \geq 3)$.

Proof

Let T be rooted at u with $\gamma_{cs}^{rs}(T) = \Delta(T) + 1$. Let $\Delta(T) = p$ and also consider u with degree p. Then $N(u) = \{u_1, u_2, \ldots\ldots, u_p\}$. Let T_j be the subtree for $j \in \{1,2, \ldots\ldots, p\}$ of T formed by u_j and its descendants. Let D be a smallest CSRSD- set with u in D. Therefore $|D \cap V(T_j)| = 1$ for $1 \leq j \leq p$ and $\beta_0(T_j) = 1$(where β_0 is an independence number of T_j). Thus $T_j = P_1$ or P_2. Hence T is a tree attained by sub-dividing atmost once each edge of $K_{1,p-1}$ $(p \geq 3)$. Converse is simple and it's straightforward.

Proposition 3.3

For any G, the $\gamma_{cs}^{rs}(G) \leq n - \gamma(G)$.

Proof

Let us consider D as a smallest CSRSD-set of G. Then $V - D$ is a D-set of G. Hence $\gamma(G) \leq |V - D| = n - \gamma_{cs}^{rs}(G)$.

Proposition 3.4

For any G, the $\gamma_{cs}^{rs}(G) \leq \frac{n\Delta(G)}{\Delta(G)+1}$.

Proof:

By the theorem 1.1[2] and the proposition 3.3, $\left\lceil \frac{n}{\Delta(G)+1} \right\rceil \leq \gamma(G) \leq n - \gamma_{cs}^{rs}(G)$. Hence $\gamma_{cs}^{rs}(G) \leq \frac{n\Delta(G)}{\Delta(G)+1}$.

IV CONCULSION

We introduced and investigated the new parameter, CSRSD-set in this paper and obtained the $\gamma_{cs}^{rs}(G)$ for some standard graphs and obtained its sharp bounds. The lower bound equality holds for complete graph and upper bound equality holds for star graph. Also, we characterize the trees for $\gamma_{cs}^{rs}(G) = m - 2$ and $\Delta(T) + 1$. Further investigations can be done by characterizing the graph G such that $\gamma_{cs}^{rs}(G) = \gamma_{cs}(G)$, $\gamma_{cs}^{rs}(G) = \gamma_s(G)$.

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