Star Chromatic Number of the Edge Corona of Two Graph

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Abstract

Graph products are introduced to obtain information on large graphs from similar information on smaller graphs. One of the most prominent graph products is Corona product of graphs. A more applicable version of it called as the edge corona was introduced ten years ago. A star coloring of an undirected graph G is an allocation of colors to every element of vertex set of the graph G such that no path of order four in G isbicolored. The star chromatic number of G is the minimum colors required to star color the graph G. Nowadays, the concept of star coloring is very much useful in tracebacking the IP addresses in networks. In this paper, we establish the exact value of the star chromatic number of edge corona product of path graph with complete graph, cycle graph, star graph, complete bipartite graph and any simple graph. Also, we have found the same for edge corona product of star graph with path graph, cycle graph, complete and any simple graph.

Keywords: *Star coloring, corona graphs, edge corona.*

1 Introduction

The graphs studied in this paper are simple, finite and undirected. The notion of star chromatic number were first initiated by Branko Grünbaum in the year 1973[8]. A proper vertex coloring of a graph G is termed as star coloring of G, if no path on four vertices in G is 2-colored. In addition to this, the induced subgraphs obtained by the vertices of any two colors is a collection of star graphs[1, 5, 8]. A star graph [9] is a complete bipartite graph of order n + 1 and it is denoted by K_(1,n). The least number of colors needed to star color the graph G is the star chromatic number of G. Fertin et al. [5] has given the star chromatic number of different graph families namely cycles, complete bipartite graphs, trees, outer planar graphs and two dimensional grids. Also, they investigated and gave bounds for the star chromatic number of other graph families namely hypercubes, tori, d-dimensional grids, planar graphsand graphs with bounded treewidth.

In [1] Albertson et al. proved that even when G is both planar and bipartite graph, the problem of determining whether G has a star coloring with 3 colors is NP - complete. The problem of finding an optimal star coloring is NP-hard and remains so even for bipartite graphs [3, 4]. Gebremedhin et al. [7] gave some works related to applications and algorithmic approach on star coloring of graphs. The concept of star coloring of graphs G is usedto traceback the IP address of the attacker withleastcount of packets [15]. While tracing back the IP addresses, the routers arerecognisedwith pre-assigned colors.

Harary and Frucht [6, 9], introduced and studied the corona product of two graphs. The corona product is neither commutative nor associative. The idea of corona graph is used to represent chemical compounds in Chemistry [11] and application of this concept include robotic navigation in networks. When the system does not resemble the structure of corona product of graphs then every time one has to separate the system with binary conflict relations into equal or almost equal conflict-free subsystems [12]. In [14], the star chromatic number was determined for the vertex corona product of graphs. In [10], Hou et al. introduced another type of corona product called the edge corona product. It is easy to observe that edge corona product of graphs can be applied in routing functions, transportation networks based on flights, trains, roads and ships in order to reach the destination and information seekers navigating in an information networks as they contain multiple paths and Hamiltonian paths. Graph products and their interesting applications have been discussed in [13].

Motivated by the above works, in this paper we study the star chromatic number of the edge corona product of path graph withnvertices and complete graph of order n (P_n◊K_n), cycle graph of same order (P_n◊C_n), star graph of n+1 vertices (P_n◊K_(1,n)), complete bipartite graph of order n_1+n_2 (P_n◊K_(n_1,n_2)), simple graph of any order(P_n◊G) and star graph on n+1 vertices with path on n vertices (K_(1,n)◊P_n), cycle graph on n vertices (K_(1,n)◊C_n), complete graph on n vertices (K_(1,n)◊K_n), simple graph of any order(K_(1,n)◊G).

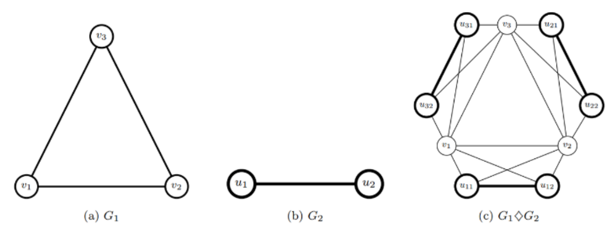
2 Preliminaries

In this section, we recall the definition of edge corona product of graphs together with some theorems given in [5, 10]. The basic graph theory terminologies that are used in this paper can be found in [2, 9].

The edge corona of two graphs G_1 with vertices (or nodes) v_1,⋯,v_n and edges e_1,⋯,e_k and G_2 is obtained by taking k copies of G_2 and for every edge e_m=v_i v_j of G_1, joining edges between the end vertices v_i,v_j of e_m with every vertex of the m^th copy of G_2[10] .

Let G_1 be the cycle graph of order 3 (see Fig. 1(a)) and G_2 be the path graph of order 3 (see Fig. 1(b)). The edge corona G_1◊G_2 of these two graphs is shown in Fig. 1(c).

Fig 1. Edge corona of C_3 and P_2



Next we recall two useful results:
Theorem 2.1[5] If C_n is a cycle with $n \geq 3$ vertices, then

$$\chi_s(C_n) = \begin{cases} 4 & when\ n = 5 \\ 3 & otherwise. \end{cases}$$

Theorem 2.2[5] Let $K_{n,m}$ be the complete bipartite graph. then

$$\chi_s(K_{n,m}) = \min\{n, m\} + 1.$$

3 Main Results

We study the star chromatic number of some edge corona product of graphs in this section.

Theorem 3.1 For any $n \geq 4$,

$$\chi_s(P_n \Diamond K_n) = n + 3.(1)$$

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and

$V(K_n) = \{u_1, u_2, \cdots, u_n\}$. Let $V(P_n \Diamond K_n) = \{v_x | 1 \leq x \leq n\} \cup \{u_{yz} | 1 \leq y \leq n-1; 1 \leq z \leq n\}$. As in the definition of the edge corona graph, end vertices $v_x, v_{x+1} \in V(P_n)$ are adjacent to every vertex in $\{u_{yz}: 1 \leq y \leq n-1; 1 \leq z \leq n\}$.

The star coloring for the vertices of $P_n \Diamond K_n$ with $n+3$ colors is given as follows:

1. For every $x \in \{1, 2, \cdots, n\}$, assign the color c_x to v_x.
2. For $1 \leq y \leq n-1$ and $1 \leq z \leq n$,
   - If $y + z + 1 \leq n + 3$, then assign the color c_{y+z+1} to u_{yz}.
   - If $y + z + 1 > n + 3$, then assign the coloring as star chromatic as given below:

* c_1 to u_{yz} when $y + z \equiv 0 (\text{mod} n + 3)$.

* c_2 to u_{yz} when $y + z \equiv 1 (\text{mod} n + 3)$.

* ... ...

* c_{n+1} to u_{yz} when $y + z \equiv n \ (\text{mod} n + 3)$.

Therefore, $\chi_s(P_n \Diamond K_n) \leq n + 3$.

To prove $\chi_s(P_n \Diamond K_n) \geq n + 3$, let us, on the contrary, assume that $\chi_s(P_n \Diamond K_n)$ is less than $n+3$, say $\chi_s(P_n \Diamond K_n) = n + 2$. Now, assign $n+2$ colors to the vertices $\{v_1, v_2, u_{yz}: 1 \leq z \leq n\}$ for proper star coloring. Since $\{v_1, v_2, u_{yz}: 1 \leq z \leq n\}$ induces a clique of order $n+2$ (say K_{n+2}), star color the clique of order n induced by the second copy of K_n, $\{u_{2z}: 1 \leq z \leq n\}$, with already existing colors such that $c(v_2) \neq c(u_{2z})$. By assigning the same $n+2$ colors to the verties of another clique induced by the third copy of K_n , $\{v_3, v_4, u_{3z}: 1 \leq z \leq n\}$,results the contradiction that one of the path of order four between these cliques is bicolored. Therefore a star coloring with $n+2$ colors is not possible. Thus, $\chi_s(P_n \Diamond K_n) \geq n + 3$. Hence, $\chi_s(P_n \Diamond K_n) = n + 3$.

Note 1. For $n = 2, 3$, $\chi_s(P_n \Diamond K_n) = n + 2$.

Example 3.1 Substitute $n = 4$ in Eqn. (1). By Theorem 3.1, we have $\chi_s(P_4 \Diamond K_4) = 4 + 3 = 7$ (Fig. 2).

Now assign the star coloring as follows:

$c(v_1) = c(u_{34}) = c_1; c(v_2) = c_2$

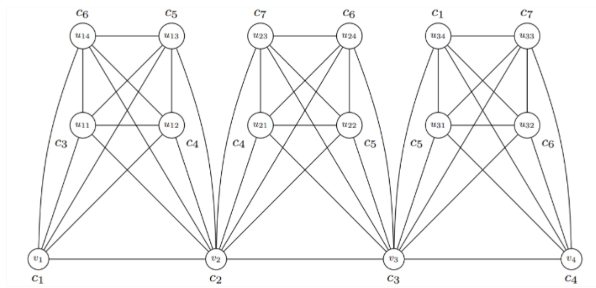
$c(v_3) = c(u_{11}) = c_3; c(v_4) = c(u_{12}) = c(v_{21}) = c_4;$

$c(u_{13}) = c(u_{22}) = c(u_{31}) = c_5;$

$c(u_{14}) = c(u_{23}) = c(u_{32}) = c_6;$

$c(u_{24}) = c(u_{33}) = c_7.$

Fig 2. $\boldsymbol{\chi_s(P_4 \Diamond K_4)}$



Theorem 3.2 For any $n \geq 4$,

$$\chi_s(P_n \Diamond C_n) = \begin{cases} 7 & n = 5 \\ 6 & otherwise. \end{cases} \quad (2)$$

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$, $V(C_n) = \{u_1, u_2, \dots, u_n\}$ and $V(P_n \Diamond C_n) = \{v_x: 1 \leq x \leq n\} \cup \{u_{yz}: 1 \leq y \leq n-1; 1 \leq z \leq n\}$. As in the definition of edge corona graph, the end vertices $v_x, v_{x+1} \in V(C_n)$ are adjacent to every vertex in$\{u_{yz}: 1 \leq y \leq n-1; 1 \leq z \leq n\}$.

Case (i): $n = 5$, Assign the following 7 colors for $P_n \Diamond C_n$ as star chromatic:

• For every $x \in \{1, \dots, 5\}$, assign the color c_x to v_x.

• If $z = 2,3,4$, then assign the color c_{z+1} to u_{1z}.

• If $z = 3,4,5$, then assign the color c_{z+1} to u_{2z}.

• If $z = 4,5$, then assign the color c_{z+1} to u_{3z}.

• If $z = 3,5$, then assign the color c_6 to u_{4z}.

• If $1 \leq y \leq 4, 1 \leq z \leq 5$, then assign the color c_7 to $u_{yz} \forall y = z$.

Assign the color c_1 to the vertices u_{31} and u_{41}. For the vertices $u_{15}, u_{21}, u_{32}, u_{42}$, assign the colors c_3 , c_4 , c_6 , c_2 , respectively. Thus $\chi_s(P_n \Diamond C_n) \leq 7$.

To prove $\chi_s(P_n \Diamond C_n) \geq 7$, let us assume that $\chi_s(P_n \Diamond C_n) < 7$, say $\chi_s(P_n \Diamond C_n) = 6$. By Theorem 2.1, assign six colors to the vertices of the set$\{v_1, v_2, u_{yz}: 1 \leq z \leq n\}$, since $\{u_{yz}: 1 \leq z \leq n\}$ is a cycle of order 5, it needs four distinct colors to have proper star coloring, v_1 and v_2 is adjacent to every vertex from $\{u_{yz}: 1 \leq z \leq n\}$. Star color the vertices of order 5 of the second copy of C_n with any of the four existing colors such that $c(v_2) \neq c(u_{2z})$. By assigning the same six colors for the another set of vertices $\{v_3, v_4, u_{3z}: 1 \leq z \leq n\}$results a contradiction that one of the path of order four between these set of vertices is bicolored. Thus, $\chi_s(P_n \diamondsuit C_n) \geq 7$.

Case (ii): $n \neq 5$. Assign the following 6 colors as star chromatic for $p_n \diamondsuit C_n$:

- For every $x \in \{1,2, \dots, 6\}$, assign the color c_x to v_x.
- For every $x \in \{7,8, \dots, n\}$, assign the color c_k, $1 \leq k \leq 6$ to all such vertices v_x that $x \equiv k \bmod 6$.
- color all the vertices of $V(C_n^{(1)}), V(C_n^{(7)}), V(C_n^{(13)})$, ... with colors c_3, c_4, c_5, respectively.
- color all the vertices of $V(C_n^{(2)}), V(C_n^{(8)}), V(C_n^{(14)})$, ... with colors c_4, c_5, c_6, respectively.
- color all the vertices of $V(C_n^{(3)}), V(C_n^{(9)}), V(C_n^{(15)})$, ... with colors c_1, c_5, c_6, respectively.
- color all the vertices of $V(C_n^{(4)}), V(C_n^{(10)}), V(C_n^{(16)})$, ... with colors c_1, c_2, c_6, respectively.
- color all the vertices of $V(C_n^{(5)}), V(C_n^{(11)}), V(C_n^{(17)})$, ... with colors c_1, c_2, c_3, respectively.
- color all the vertices of $V(C_n^{(6)}), V(C_n^{(12)}), V(C_n^{(18)})$, ... with colors c_2, c_3, c_4, respectively.

Therefore $\chi_s(P_n \diamondsuit C_n) \leq 6$. To prove $\chi_s(P_n \diamondsuit C_n) \geq 6$, let us assume that $\chi_s(P_n \diamondsuit C_n) < 6$, say $\chi_s(P_n \diamondsuit C_n) = 5$. By Theorem 2.1, assign five colors to the vertices $\{v_1, v_2, u_{yz}: 1 \leq z \leq n\}$, since $\{u_{yz}: 1 \leq z \leq n\}$ is a cycle, v_1 and v_2 are adjacent to each $\{u_{yz}: 1 \leq z \leq n\}$. It is possible to star color the second copy of C_5 with any 3 of the existing colors such that $c(v_2) \neq c(u_{yz})$. Assigning the same five colors to another set of vertices $\{v_3, v_4, u_{3z}: 1 \leq z \leq n\}$results a contradiction that one of the paths of order four between these set of vertices is bicolored. Therefore a star coloring with five colors is not possible. Thus $\chi_s(P_n \diamondsuit C_n) \geq 6$. Hence $\chi_s(P_n \diamondsuit C_n) = 6$ for $n \neq 5$.

Note 2. For $n = 3$, $\chi_s(P_n \diamondsuit C_n) = 5$.

The cases considered in Theorem 3.2 are demonstrated in Example 3.2.

Example 3.2 Case (i): If $n = 5$ in Eqn. (2), then $\chi_s(P_5 \diamondsuit C_5) = 7$.

Assign the star coloring for $\chi_s(P_5 \diamondsuit C_5)$ as follows:

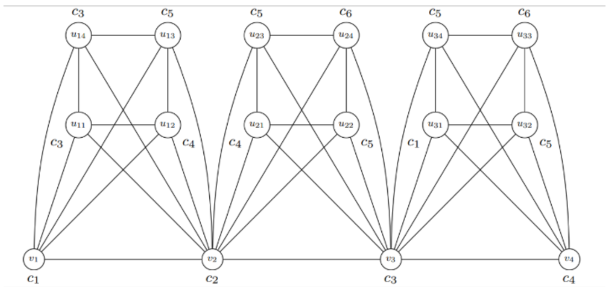
$c(v_1) = c(u_{31}) = c(u_{41}) = c_1; c(v_2) = c(u_{42}) = c_2;$
$c(v_3) = c(u_{12}) = c_3; c(v_4) = c(u_{13}) = c(u_{23}) = c_4;$
$c(v_5) = c(u_{14}) = c(u_{24}) = c(u_{34}) = c_5;$
$c(u_{15}) = c(u_{25}) = c(u_{35}) = c(u_{45}) = c_6;$
$c(u_{11}) = c(u_{22}) = c(u_{33}) = c(u_{44}) = c(u_{55}) = c_7.$

It is obvious that the graph $P_5 \diamondsuit C_5$ accepts a star coloring.

Case (ii): Let $n \neq 5$. Let $n = 4$ in Eqn. (2). Then $\chi_s(P_4 \diamondsuit C_4) = 6$ (See Fig. 3). Assign the star coloring as follows:

$c(v_1) = c(u_{31}) = c_1; c(v_2) = c_2;$
$c(v_3) = c(u_{31}) = c_3; c(v_4) = c(u_{12}) = c(u_{21}) = c_4;$
$c(u_{13}) = c(u_{22}) = c(u_{31}) = c_5;$
$c(u_{21}) = c(u_{23}) = c(u_{32}) = c(u_{34}) = c_6.$

Fig 3. $\chi_s(P_4 \diamondsuit C_4)$



Theorem 3.3 Let $n \geq 4$ be a positive integer. Then

$$\chi_s(P_n \diamondsuit K_{1,n}) = 5. \qquad (3)$$

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $V(K_{1,n}) = \{u_y, u_{yz} : 1 \leq y \leq n-1; 1 \leq z \leq n\}$, and $V(P_n \diamondsuit K_{1,n}) = \{v_x / 1 \leq x \leq n\} \cup \{u_y : 1 \leq y \leq n-1\} \cup \{u_{yz} / 1 \leq y \leq n; 1 \leq z \leq n\}$. As in the definition of edge corona graph, all pairs of end vertices $v_x, v_{x+1} \in V(P_n)$ is adjacent to every vertex in $\{u_y : 1 \leq y \leq n-1\} \cup \{u_{yz} : 1 \leq y \leq n-1; 1 \leq z \leq n\}$.

- For every $x \in \{1, 2, \dots, n\}$, color the vertices v_x with colors c_1, c_2, c_3, c_4, c_5, respectively.
- For every $y \in \{1, 2, \dots, n-1\}$, color the vertices u_y with colors c_3, c_4, c_5, c_1, c_2, respectively.
- For $1 \leq y \leq 5$; $1 \leq z \leq n$, color the vertices u_{yz} with colors c_4, c_5, c_1, c_2, c_3, respectively.
- For $6 \leq y \leq n$ and $1 \leq z \leq n$, color the vertices u_{yz} as a star coloring as follows:
  - If $y \equiv 1 \bmod 5$, then color the vertices u_{yz} with color c_4.
  - If $y \equiv 2 \bmod 5$, then color the vertices u_{yz} with color c_5.
  - If $y \equiv 3 \bmod 5$, then color the vertices u_{yz} with color c_1.
  - If $y \equiv 4 \bmod 5$, then color the vertices u_{yz} with color c_2.
  - If $y \equiv 0 \bmod 5$, then color the vertices u_{yz} with color c_3.

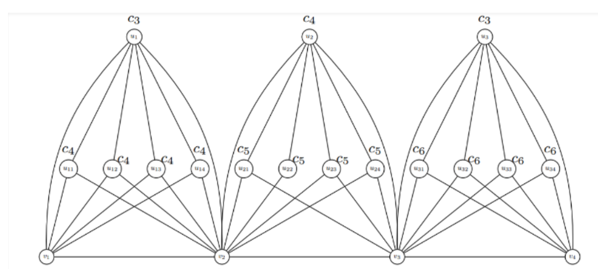
Therefore $\chi_s(P_n \diamondsuit K_{1,n}) \leq 5$. To prove $\chi_s(P_n \diamondsuit K_{1,n}) \geq 5$. Let us assume that $\chi_s(P_n \diamondsuit K_{1,n}) < 5$, say $\chi_s(P_n \diamondsuit K_{1,n}) = 4$. Assign four colors to $\{v_1, v_2, u_1, u_{1z} : 1 \leq z \leq n\}$. Since $\{u_1, u_{1z} : 1 \leq z \leq n\}$ is a star graph, it needs two distinct colors and each $\{u_1, u_{1z} : 1 \leq z \leq n\}$ is adjacent to v_1 and v_2. This shows that v_1 and v_2 need another two distinct colors for proper star coloring. Star color the second copy of $K_{1,n}$ (i.e. $\{u_2, u_{2z} : 1 \leq z \leq n\}$) with any two of colors which already exist such that $c(v_2) \neq c(u_2) \neq \{c(u_{2z})\ ;\ 1 \leq z \leq n\}$. By assigning the same four colors to another set of vertices $\{v_3, v_4, u_3, u_{3z} : 1 \leq z \leq n\}$, results a contradiction that one the path of order four between these set of vertices is bicolored. Therefore a star coloring with 4 colors is not possible. Thus $\chi_s(P_n \diamondsuit K_{1,n}) \geq 5$. Hence $\chi_s(P_n \diamondsuit K_{1,n}) = 5$.

Note 3. For $n = 2, 3$, $\chi_s(P_n \diamondsuit K_{1,n}) = 4$.

Example 3.3 Substitute $n = 4$ in Eqn. (3). By Theorem 3.3, we observe that $\chi_s(P_4 \diamondsuit K_{1,4}) = 5$ (see Fig. 4), which can be done by assigning the colors in the following way:

$c(v_1) = c(u_3) = c_1; c(v_2) = c_2;$
$c(v_3) = c(u_1) = c_3;$
$c(u_{11}) = c(u_{12}) = c(u_{13}) = c(u_{14}) = c_4;$
$c(u_{21}) = c(u_{22}) = c(u_{23}) = c(u_{24}) = c_5;$
$c(u_{31}) = c(u_{32}) = c(u_{33}) = c(u_{34}) = c_6.$

Fig 4. $\chi_s(P_4 \diamondsuit K_{1,4})$



Theorem 3.4 For $n \geq 4$ and $n = n_1$ or $n = n_2$,

$$\chi_s(P_n \diamondsuit K_{n_1, n_2}) = \min\{n_1, n_2\} + 4. \qquad (4)$$

Proof. Case (i): If $n_1 \leq n_2$, let $n =$

$\max\{n_1, n_2\} = n_2$. Let $V(P_n) = \{v_x: 1 \leq x \leq n_2 - 1\}$, $V(K_{n_1,n_2}) = \{u_{yz}: 1 \leq y \leq n_2 - 1; 1 \leq z \leq n_1\} \cup \{w_{yz}: 1 \leq y \leq n_2 - 1; 1 \leq z \leq n_2\}$ and $V(P_n \Diamond K_{n_1,n_2}) = \{v_x: 1 \leq x \leq n\} \cup \{u_{yz}: 1 \leq y \leq n_2 - 1; 1 \leq z \leq n_1\} \cup \{w_{yz}: 1 \leq y \leq n_2 - 1; 1 \leq z \leq n_2\}$. As in the definition of edge corona graph, all end vertices $v_x, v_{x+1} \in V(P_n)$ are adjacent to each vertex in $\{u_{yz}: 1 \leq y \leq n_2 - 1; 1 \leq z \leq n_1\} \cup \{w_{yz}: 1 \leq y \leq n_2 - 1; 1 \leq z \leq n_2\}$. Assign the star coloring as follows:

• For $1 \leq x \leq n-1$, if $x \leq n_1 + 4$, assign the color c_x to the vertex v_x.

• For $1 \leq x \leq n-1$, color the vertex v_x with c_k if $x > n_1 + 4$,
if $x \equiv k (\bmod n_1 + 4); 1 \leq k \leq (n_1 + 4)$.

• For $1 \leq y \leq n_2 - 1, 1 \leq z \leq n_1$, color the vertex u_{yz} with c_{y+z+1} if $y + z \leq n_1 + 4$.

• For $1 \leq y \leq n_2 - 1; 1 \leq z \leq n_1$, color the vertex u_{yz} with c_k if $y + z \equiv 0 (\bmod n_1 + 4)$.

• For $1 \leq y \leq 2; 1 \leq z \leq n_2$, if c_k is the color of the vertex u_{yn_1} then color all the vertices w_{yz} with color c_{k+1}.

• For $3 \leq y \leq n_2 - 1; 1 \leq z \leq n_2$, color the vertices w_{yz} with one of the pre-assigned colors such that $c(w_{yz}) \neq c(v_x) \neq c(v_{x-1}) \neq c(u_{yz})$.

Thus $\chi_s(P_n \Diamond K_{n_1,n_2}) \leq n_1 + 4$ when $n_1 < n_2$. To prove $\chi_s(P_n \Diamond K_{n_1,n_2}) \geq n_1 + 4$, let us assume that $\chi_s(P_n \Diamond K_{n_1,n_2}) < n_1 + 4$, say $\chi_s(P_n \Diamond K_{n_1,n_2}) = n_1 + 3$. By Theorem 2.2, $n_1 + 1$ colors needed for a proper star coloring to the set $\{u_{1z}: 1 \leq z \leq n_1; w_{1z}: 1 \leq z \leq n_2\}$. Since the vertices v_1 and v_2 are adjacent to each of the vertices in the set $\{u_{1z}: 1 \leq z \leq n_1\} \cup \{w_{1z}: 1 \leq z \leq n_2\}$, we need $n_1 + 3$ colors to star color $\{v_1, v_2, u_{1z}: 1 \leq z \leq n_1\} \cup \{w_{1z}: 1 \leq z \leq n_2\}$. Star color the second copy of K_{n_1,n_2} with colors which already exist such that $c(v_2) \neq c(u_{ij}) \neq c(w_{ij})$. Assigning the same $n_1 + 3$ colors to the set $\{v_3, v_4, u_{3z}: 1 \leq z \leq n_1, w_{3z}: 1 \leq z \leq n_2\}$results a contradiction that one of the path of order four between these set of vertices is bicolored. Thus $\chi_s(P_n \Diamond K_{n_1,n_2}) = n_1 + 3$ is not possible. Therefore $\chi_s(P_n \Diamond K_{n_1,n_2}) \geq n_1 + 4$. Hence $\chi_s(P_n \Diamond K_{n_1,n_2}) = n_1 + 4$.

Case (ii): If $n_2 < n_1$. Let $n = \max\{n_1, n_2\} = n_1$. Let $V(P_n) = \{v_x: 1 \leq x \leq n_2 - 1\}$, $V(K_{n_1,n_2}) = \{w_{yz}: 1 \leq y \leq n_1 - 1; 1 \leq z \leq n_2\} \cup \{u_{yz}: 1 \leq y \leq n_1 - 1; 1 \leq z \leq n_1\}$ and $V(P_n \Diamond K_{n_1,n_2}) = \{v_x: 1 \leq x \leq n\} \cup \{w_{yz}: 1 \leq y \leq n_1 - 1; 1 \leq z \leq n_2\} \cup \{u_{yz}: 1 \leq y \leq n_1 - 1; 1 \leq z \leq n_1\}$. As in the definition of edge corona graph, all end vertices $v_x, v_{x+1} \in V(P_n)$ are adjacent to all the vertices from the set $\{w_{yz}: 1 \leq y \leq n_1 - 1; 1 \leq z \leq n_2\} \cup \{u_{yz}: 1 \leq y \leq n_1 - 1; 1 \leq z \leq n_1\}$. Assign the star coloring as follows:

• For $1 \leq x \leq n-1$, assign the color c_x to the vertex v_x if $x \leq n_2 + 4$.

• For $1 \leq x \leq n-1$, color the vertex v_x with c_k if $x > n_2 + 4$;
if $x \equiv k (\bmod n_2 + 4); 1 \leq k \leq (n_2 + 4)$.

• For $1 \leq y \leq n_1 - 1, 1 \leq z \leq n_2$, color the vertex w_{yz} with c_{y+z+1} if $y + z \leq n_2 + 4$.

• For $1 \leq y \leq n_1 - 1; 1 \leq z \leq n_2$, color the vertex w_{yz} with c_k if $y + z \equiv 0 (\bmod n_2 + 4)$.

• For $1 \leq y \leq 2; 1 \leq z \leq n_1$, if c_k is the color of the vertex u_{yn_2} then color all the vertices u_{yz} with color c_{k+1}.

• For $3 \leq y \leq n_1 - 1; 1 \leq z \leq n_1$, color the vertices u_{yz} with one of the pre-assigned colors such that $c(u_{yz}) \neq c(v_x) \neq c(v_{x-1}) \neq c(w_{yz})$.

Thus $\chi_s(P_n \Diamond K_{n_1,n_2}) \leq n_2 + 4$ when $n_2 < n_1$. To prove $\chi_s(P_n \Diamond K_{n_1,n_2}) \geq n_2 + 4$, let us assume that $\chi_s(P_n \Diamond K_{n_1,n_2}) < n_2 + 4$, say $\chi_s(P_n \Diamond K_{n_1,n_2}) = n_2 + 3$. By Theorem 2.2, $n_2 + 1$ colors needed for a proper star coloring to the set $\{w_{1z}: 1 \leq z \leq n_2\} \cup \{u_{1z}: 1 \leq z \leq n_1\}$. Since the vertices v_1 and v_2 are adjacent to each of the vertices in $\{w_{1z}: 1 \leq z \leq n_2\} \cup \{u_{1z}: 1 \leq z \leq n_1\}$, it needs $n_2 + 3$ colors to star color $\{v_1, v_2, w_{1z}: 1 \leq z \leq n_2, u_{1z}: 1 \leq z \leq n_1\}$. Star color the second copy of K_{n_1,n_2}

with existing colors such that $c(v_2) \neq c(w_{yz}) \neq c(u_{yz})$. By assigning the same $n_2 + 3$ colors to the set $\{v_3, v_4, w_{3y}: 1 \leq y \leq n_2, u_{3y}: 1 \leq y \leq n_1\}$ results a contradiction that onee of the path of order four between these set of vertices is bicolored. Thus $\chi_s(P_n \Diamond K_{n_1,n_2}) = n_2 + 3$ is not possible. Therefore $\chi_s(P_n \Diamond K_{n_1,n_2}) \geq n_2 + 4$. Hence $\chi_s(P_n \Diamond K_{n_1,n_2}) = n_2 + 4$.

Note 4. For $n = 2, 3$ and $n = n_1$ or $n = n_2$,

$$\chi_s(P_n \Diamond K_{n_1,n_2}) = \min\{n_1, n_2\} + 3.$$

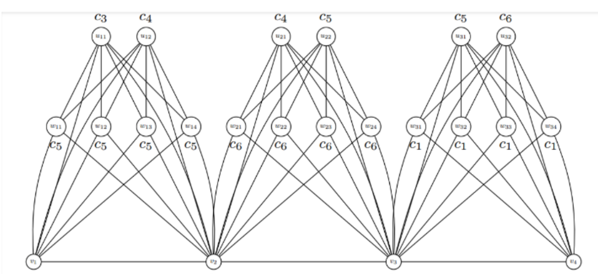
The cases considered in Theorem 3.4 are demonstrated in Example 3.4.

Example 3.4 Case (i): Let $n_1 < n_2$ and let $n = n_2 = 4\ and\ n_1 = 2$ in Eqn. (4). Then $\chi_s(P_4 \Diamond K_{2,4}) = \min\{2,4\} + 4 = 2 + 4 = 6$ (see Fig. 4). Assign the star coloring as follows:

$$c(v_1) = c(w_{31}) = c(w_{32}) = c(w_{33}) = c(w_{34}) = c_6, c(v_2) = c_2;$$
$$c(v_3) = c(u_{11}) = c_3; c(v_4) = c(u_{12}) = c_4;$$
$$c(w_{11}) = c(w_{12}) = c(w_{13}) = c(w_{14}) = c(u_{22}) = c(u_{31}) = c_5;$$
$$c(w_{21}) = c(w_{22}) = c(w_{23}) = c(w_{24}) = c(u_{31}) = c_6.$$

Case (ii): Let $n_1 \geq n_2$. By Theorem 3.2 of the Case (ii), it is easy to show that $\chi_s(P_n \Diamond K_{n_1,n_2}) = \min\{n_1, n_2\} + 4$.

Fig 5. $\boldsymbol{\chi_s(P_4 \Diamond K_{2,4})}$



Notation. Let G be a simple graph of any order say t and denote the vertices of each copy of G as u_{yz} where, y represents the corresponding copy of G and z ranges from 1 to t. We use this notations in Theorem 3.5 and Theorem 3.9.

Theorem 3.5 For any $n \geq 4$,

$$\chi_s(P_n \Diamond G) = \chi_s(G) + 3. \quad (5)$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $\chi_s(G) = r$.

Color the vertices of $P_n \Diamond G$ with $r + 3$ colors as follows and it is denoted by σ.

- Color the vertex v_x with color c_k such that, $x \equiv k(\text{mod}r + 3)$ for each $x \in \{1,2, \ldots, n\}$ and $0 \leq k \leq r + 3$.
- Assign the star coloring to all the copies of G in $P_n \Diamond G$ with the color set $\{c_k / y + z + 1 \equiv k(\text{mod}r + 3);\ 1 \leq y \leq n - 1;\ 0 \leq k \leq r + 3\}$.

It is obvious that σ is a proper coloring. Now, we have to prove that σ is a star coloring.

Let P_4 be any path of order four in $P_n \Diamond G$. If $|V(P_4) \cap V(P_n)| = 4$ or 0 then it is obvious that, P_4 is not bicolored. If $|V(P_4) \cap V(P_n)| = 3$ then P_4 is not bicolored, since, any three successive vertices of P_n in $P_n \Diamond G$ have three distinct colors. If $|V(P_4) \cap V(P_n)| = 2$ either P_4 has two vertices on each copy of G of $P_n \Diamond G$ or it has two vertices on two different copies of G say G_i and G_j of $P_n \Diamond G$. For the former case, let $\sigma(v_x) = c_k;\ x \equiv k(\text{mod}r + 3), \sigma(v_{x+1}) = c_m;\ x + 1 \equiv m(\text{mod}r + 3)$ and $c_k \neq c_m \neq c_l$ where $x + y \equiv l(\text{mod}r + 3)$ then P_4 has atleast three vertices with different colors. Therefore P_4 is not bicolored. For the latter case, $\sigma(v_{x+1}) = c_m;\ x + 1 \equiv m(\text{mod}r + 3)$, $\sigma(v_{x+2}) = c_p;\ x + 2 \equiv p(\text{mod}r + 3)$, any vertex of G_j has color c_{x+y+1}, obviously, three sucessive vertices have dstinct color, i.e., P_4 is not bicolored. If $|V(P_4) \cap V(P_n)| = 1$, since the number of colors on the three successive vertices is atleast two and $\sigma(v_x)$ is different from other color then P_4 is not bicolored. Thus, $\chi_s(P_n \Diamond G) \leq \chi_s(G) + 3$.

To prove $\chi_s(P_n \Diamond G) \geq \chi_s(G) + 3$, on the contrary, let us assume that $\chi_s(P_n \Diamond G)$ is less than $r + 3$, say $\chi_s(P_n \Diamond G) = r + 2$. Assign $r + 2$ colors to the vertices $\{v_1, v_2, u_{yz}: 1 \leq y \leq n - 1$ and z starts from 1 to t$\}$ (u_{yz} refers the label of vertices of copies of G) for proper star coloring. Since G has star chromatic number $r + 2$, star color the next copy of G with r colors. By assigning the same $r + 2$ colors to another copy of G with same $r+2$ colors, results a contradiction that one of the path of order four between these vertes sets is bicolored. Therefore a star coloring with $r+2$ colors is not possible. Thus, $\chi_s(P_n \Diamond G) \geq r+3$. Hence, $\chi_s(P_n \Diamond G) = \chi_s(G)+3$.

Note 5. For $n = 2, 3$,

$$\chi_s(P_n \Diamond G) = \chi_s(G)+2.$$

Theorem 3.6 Let $n \geq 3$ be any positive integer,

$$\chi_s(K_{1,n} \Diamond P_n) = 5. \tag{6}$$

Proof. Let $V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Let $V(K_{1,n} \Diamond K_n) = \{v_x : 1 \leq x \leq n+1\} \cup \{u_{yz} : 1 \leq y \leq n; 1 \leq z \leq n\}$. As in the definition of edge corona graph, all end vertices $v_x, v_{x+1} \in V(K_{1,n})$ are adjacent to each vertex in $\{ u_{yz} : 1 \leq y \leq n; 1 \leq z \leq n\}$. Assign the following 5 colors as star chromatic for $K_{1,n} \Diamond P_n$:

- Assign color c_5 to the vertex v_1.
- For every $x \in \{2,3,\cdots,n+1\}$, color the vertices v_x with color c_4.
- Assign the colors c_1, c_2, c_3 respectively to all the vertices of each copy of P_n (i.e. $\forall\{u_{yz} : 1 \leq y \leq n; 1 \leq z \leq n\}$).

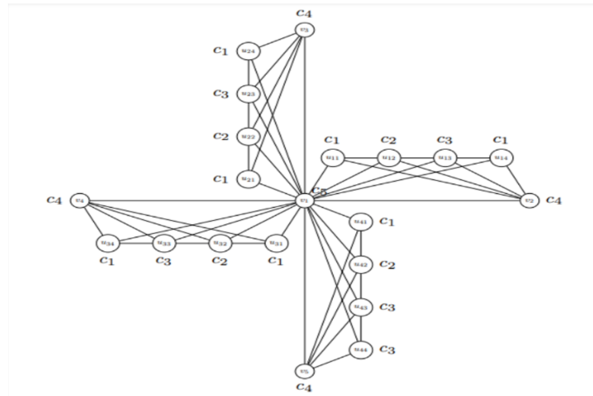
Thus $\chi_s(K_{1,n} \Diamond P_n) \leq 5$. To prove $\chi_s(K_{1,n} \Diamond P_n) \geq 5$. Let us assume that $\chi_s(K_{1,n} \Diamond K_n) < 5$, say $\chi_s(K_{1,n} \Diamond K_n) = 4$. We must assign three distinct colors to all of the vertices of P_n, $\{u_{yz} : 1 \leq y \leq n : 1 \leq z \leq n\}$ and we need another two distinct colors to color the vertices v_1 and v_2 as v_1 and v_2 are adjacent with all the vertices in $\{u_{yz} : 1 \leq y \leq n : 1 \leq z \leq n\}$, which is a contradiction. Thus $\chi_s(K_{1,n} \Diamond P_n) \geq 5$. Hence $\chi_s(K_{1,n} \Diamond P_n) = 5$.

Example 3.5 Substitute $n = 4$ in Eqn. (6). By Theorem 3.6, we observe that $\chi_s(K_{1,4} \Diamond P_4) = 5$, (see Fig. 6), which can be done by assigning the colors in the following way:

$$c(v_1) = c_5; c(v_2) = c(v_3) = c(v_4) = c(v_5) = c_4;$$

$c(u_{11}) = c(u_{21}) = c(u_{31}) = c(u_{41}) = c_1;$
$c(u_{12}) = c(u_{22}) = c(u_{32}) = c(u_{42}) = c_2;$
$c(u_{13}) = c(u_{23}) = c(u_{33}) = c(u_{43}) = c_3;$
$c(u_{14}) = c(u_{24}) = c(u_{34}) = c(u_{44}) = c_4.$

Fig 6. $\chi_s(K_{1,4} \Diamond P_4)$



Theorem 3.7 For any $n \geq 2$,

$$\chi_s(K_{1,n} \Diamond K_n) = n+2. \tag{7}$$

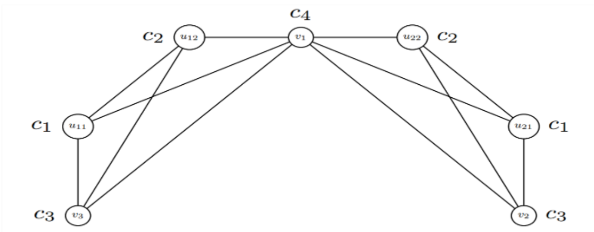
Proof. Let $V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. Let $V(K_{1,n} \Diamond K_n) = \{v_x : 1 \leq x \leq n+1\} \cup \{u_{yz} : 1 \leq y \leq n; 1 \leq z \leq n\}$. As in the definition of edge corona graph, all end vertices $v_x, v_{x+1} \in V(K_{1,n})$ are adjacent to every vertex in $\{u_{yz} : 1 \leq y \leq n; 1 \leq z \leq n\}$. Assign the following $n+2$-coloring to $K_{1,n} \Diamond K_n$ as a star chromatic coloring:

- Assign color c_{n+2} to the vertex v_1.
- For every $x \in \{2,3,\ldots,n+1\}$, assign the color c_{n+1} to v_x.
- For every $y \in \{1,2,\ldots,n\}$ and $z \in \{1,2,\ldots,n\}$, color the vertices u_{yz} with color c_z.

Thus $\chi_s(K_{1,n} \Diamond K_n) \leq n+2$. To prove $\chi_s(K_{1,n} \Diamond K_n) \geq n+2$, let us assume that $\chi_s(K_{1,n} \Diamond K_n) < n+2$, say $\chi_s(K_{1,n} \Diamond K_n) = n+1$, which is not possible since $\{v_1, v_2, u_{1z} : 1 \leq z \leq n\}$ induces a clique of order $n+2$ (say K_{n+2}). Therefore a star coloring with $n+1$ colors is not possible. Thus $\chi_s(K_{1,n} \Diamond K_n) \geq n+2$. Hence $\chi_s(K_{1,n} \Diamond K_n) = n+2$.

Example 3.6 Substitute $n = 2$ in Eqn. (7). By Theorem 3.7, we observe that $\chi_s(K_{1,2}, K_2) = 2+2 = 4$ (see Fig. 7), which can be done by assigning the colors in the following way: $c(v_1) = c_4$, $c(v_2) = c(v_3) = c_3$, $c(u_{11}) = c(u_{21}) = c_1, c(u_{12}) = c(u_{22}) = c_2$.

Fig 7. $\boldsymbol{\chi_s(K_{1,2}\Diamond K_2)}$



Theorem 3.8 For any $n \geq 3$,

$$\chi_s(K_{1,n}\Diamond C_n) = \begin{cases} 6 & n = 5 \\ 5 & otherwise \end{cases} \quad (8)$$

Proof. Let $V(K_{1,n}) = \{v_1, v_2, \cdots, v_{n+1}\}$, $V(C_n) = \{u_1, u_2, \dots, u_n\}$ and $V(K_{1,n}\Diamond C_n) = \{v_x: 1 \leq x \leq n+1\} \cup \{u_{yz}: 1 \leq y \leq n; 1 \leq z \leq n\}$. As in the definition of edge corona graph, all end vertices $v_x, v_{x+1} \in V(K_{1,n})$ are adjacent to every vertex in $\{u_{yz}: 1 \leq y \leq n; 1 \leq z \leq n\}$.

Case (i): Let $n = 5$. Assign the following 6-colors to $P_n\Diamond C_n$ as a star chromatic coloring:

- Assign color c_6 to the vertex v_1.
- For every $x \in \{2,3, \cdots, n+1\}$, assign the color c_5 to v_x.
- For every $y \in \{1,2, \cdots, n\}$ and $z \in \{1,2, \cdots n\}$, assign the colors c_1, c_2, c_3, c_4, c_2 to all the vertices of each copy of C_n of order 5.

Thus $\chi_s(K_{1,n}\Diamond C_n) \leq 6$. To prove $\chi_s(K_{1,n}\Diamond C_n) \geq 6$, let us assume that $\chi_s(K_{1,n}\Diamond C_n) = 5$, which is not possible since by Theorem 4, four distinct colors needed for a proper star coloring and each $\{u_{1z}: 1 \leq z \leq n\}$ is adjacent to v_1 and v_2. This shows that v_1 and v_2 need another two distinct colors. Therefore a star coloring with 5 colors is impossible. Thus $\chi_s(K_{1,n}\Diamond C_n) \geq 6$. Hence $\chi_s(K_{1,n}\Diamond C_n) = 6$.

Case (ii): Let $n \neq 5$. Assign the following 5-colors to $P_n\Diamond C_n$ as a star chromatic coloring:

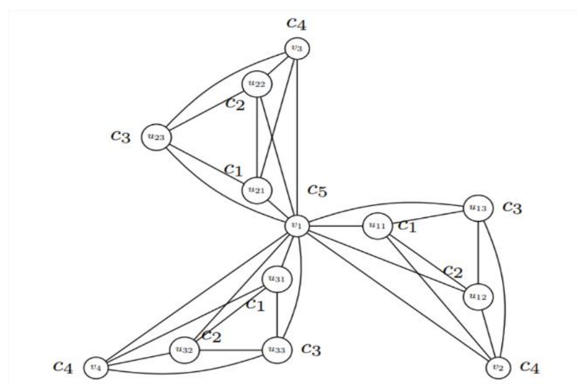
- Assign color c_5 to the vertex v_1.
- For every $x \in \{2,3, \cdots, n+1\}$, color the vertices v_x with color c_4.
- For every $y \in \{1,2, \dots, n\}$ and $z \in \{1,2, \dots, n\}$, assign the colors c_1 , c_2 , c_3 respectively to all the vertices of each copy of C_n of order n.

Thus $\chi_s(K_{1,n}\Diamond C_n) \leq 5$. To prove $\chi_s(K_{1,n}\Diamond C_n) \geq 5$, let us assume that $\chi_s(K_{1,n}\Diamond C_n) = 4$, which is not possible since by Theorem 4, three distinct colors needed for a proper star coloring and each $\{u_{1z}: 1 \leq z \leq n\}$ is adjacent to v_1 and v_2. This shows that v_1 and v_2 need another two distinct colors. Therefore a star coloring with 4 colors is not possible. Thus $\chi_s(K_{1,n}\Diamond C_n) \geq 5$. Hence $\chi_s(K_{1,n}\Diamond C_n) = 5$.

Example 3.7 Substitute $n = 3$ in Eqn. (8). By Theorem 3.8, we observe that $\chi_s(K_{1,3}, C_3) = 5$, (see Fig. 8), which can be done by assigning the colors in the following way:

$c(v_1) = c_5; c(v_2) = c(v_3) = c(v_4) = c_4;$
$c(u_{11}) = c_{u_{21}} = c(u_{31}) = c_1;$
$c(u_{12}) = c(u_{22}) = c(u_{32}) = c_2;$
$c(u_{13}) = c(u_{23}) = c(u_{33}) = c_3.$

Fig 8. $\boldsymbol{\chi_s(K_{1,3}\Diamond C_3)}$



Theorem 3.9 For any $n \geq 2$,

$$\chi_s(K_{1,n}\Diamond G) = \chi_s(G) + 2. \quad (9)$$

Proof. Let $V(K_{1,n}) = \{v_1, v_2, \dots, v_n\}$ and $\chi_s(G) = r$.

1. Assign the color c_{r+2} to the vertex v_1 and for every $x \in \{2,3, \dots, n+1\}$, assign the color c_{r+1} to v_x.
2. Assign the star coloring for all the copies

of G in $K_{1,n}\Diamond G$ with colors from the color set $\{c_k/1 \leq k \leq r\}$.
Thus, $\chi_s(K_{1,n}\Diamond G) \leq r+2$. To prove $\chi_s(K_{1,n}\Diamond G) < r+2$, say $\chi_s(K_{1,n}\Diamond G) = r+1$ which is not possible, since we need two distinct colors for the vertices of $K_{1,n}$ of $K_{1,n}\Diamond G$. Therefore, star coloring with $r+1$ color is impossible. Thus, $\chi_s(K_{1,n}\Diamond G) \geq r+2$.
Hence, $\chi_s(K_{1,n}\Diamond G) = r+2$.

4 Conclusion

The idea of corona graph is used to represent chemical compounds in Chemistry and application of this concept include robotic navigation in networks. In recent times the concept of star coloring is very useful in day to day life especially in IP traceback in networks. With this motive, in this paper, we have determined the star chromatic number of edge corona product of path graph with complete graph, cycle graph, star graph, complete bipartite graph and any simple graph. Also, we have found the same for edge corona product of star graph with path graph, cycle graph, complete and any simple graph. The future scope of this paper is to find the star chromatic number of edge corona product of various families of graphs and to find the star chromatic number of neighbourhood corona product of different families of graphs.

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