A fractional epidemiological model for prediction and simulation the outbreaks of dengue fever outbreaks in Sudan

Thwiba A. Khalid
Academy of Engineering and Medical Science, Khartoum, Sudan, tabdulrhman@bu.edu.sa
Department of mathematics, Faculty of Science and Arts, Al-baha university, Baljurashi 1988, Saudi Arabia.

Aymen Imam
Department of mathematics, Faculty of Science and Arts, Al-baha university, Baljurashi 1988, Saudi Arabia, aimam@bu.edu.sa

Abdulgader Bahatheg
Department of mathematics, Faculty of Science and Arts, Al-baha university, Baljurashi 1988, Saudi Arabia, abahathiq@bu.edu.sa

Shahinaz A. Elsamani
Department of mathematics, Faculty of Science, Bakht Al-Ruda University, White Nile state, Sudan, Shahinazel121@gmail.com

Bashir El Mukhtar
Department of Community medicine, Faculty of Medicine, University of Gadarif, Gadarif state, Sudan, bashirelnour@gmail.com

Abstract

The purpose of this study is to develop a fractional epidemiological model for dengue fever in Sudan as well as simulate the model by utilizing real data and the Adam-Bashforth and Caputo-Fabrizio operators to predict the incidence of dengue fever. The results of our model included stability analysis, reproduction numbers, and the existence and uniqueness of remedies. The simulation of the model based on real data revealed that the reproduction number is equal to 10, which indicates that Sudan is experiencing an outbreak of dengue fever. The government may receive a suggestion based on the modeling of the dengue fever prototype for the substantial proportion of dengue cases in Sudan.

Keywords: *A fractional Caputo-Fabrizio; Prediction; simulation; Dengue; Sudan.*

1. INTRODUCTION

Dengue fever (DF) is caused by a virus spread by parasitized Aides Aegyptus mosquitoes. It is a tropical and subtropical disease [1]. Yes, this is the same response. Pathogens progress to the more profound illnesses, dengue hemorrhagic fever (DHF) or septicemic (DSS) [2]. Recent years have seen an increase in the occurrence of DF, making it a public health issue of worldwide significance that necessitates the

involvement of relevant bodies for every International Health Regulation (IHR) 2005. As shown by the World Health Organization (WHO), two-fifths of the population of the earth seems to be under threat from DF sepsis, and the ailment remains prevalent in over 100 countries [3, 4]. Several more influence the diffusion and propagation of DF, and novel studies indicate that DF lesions are usually aligned with interior tendencies, such as water-storage mechanisms that create a reproductive channel for mosquitoes. Mosquitoes are also thought to spend the night inside. Malady diffusion is aided by urbanization, which efficiently delivers albedo owing to unviable prevention strategies, climate variability, and a rise in commercial airplanes; this has also assisted the slight acceleration of viremic beings and the international scatter of the ailment [5]. The virus has been swirling all across Africa ever since the 20th century. Until 1980, the serogroups' occurrences had increased exponentially [6–14]. Dengue virus is prevalent throughout Sudan, and its proportion has been reduced to the country's coastline and subcoastal territories, where the illness was first described by Saigh in 1906 and Balfour in 1907 on the Red Sea shoreline, in Port Sudan downtown [3,4,5,6]. DNF has recently swept the country. In fact, the bulk of dengue floods in the North African and Middle East were reveled in Sudan [6,7]. DNF-2 was first realized in patients admitted to hospital of an East Sudan in 1984, and it was first confirmed in Kassala region during an outbreak in 2016–17. Sudan was the initial African place to confirm DNF-1 that year [8, 9]. DNF pathogens have been noticed in individuals from bacterial loads in the east, north, south, and focal Sudan varied widely from 7 to 25% [5, 6, 3]. These pathogens have been associated with febrile disease outbreaks following heavy rain and flooding.

According to entomological scrutiny, the dominant mosquito species in the area was Aedes aegypti, which is known as the primary vector of this virus in urban settings. Later that year, refugees in Darfur, Sudan, experienced a unique dengue fever outbreak with co-circulating DNF-2 and 3 [13]. As a result, a second DENV-2 outbreak occurred in east Sudan between 2016 and 2017 [8]. According to studies, Sudan has a high prevalence of DNF infections (67%), with the coastal region having an even higher prevalence (89%). They reiterated that the bulk of In each of these regions, persons were diagnosed with multiple pathogens, and so all DNF (1-4) serogroups were prevalent [13]. DNF-1 and DNF-3 have arisen in the Darfur region of western Sudan, exacerbating a dengue fever outbreak [14]. The key determinants for DENV pathogen in Sudan were poverty, an absence of mosquito control, sleeping outside, absence of essential civic facilities these rather water delivery and water supply storage, and geographic placement [9, 15]. This paper aims to describe the dynamics of the dengue epidemic outbreak in Sudan using the fractional-order differential operator Caputo-Fabrizio (CF) and the fractional Euler method. The second goal of this research is to assess the effectiveness of these algorithms and use the most precise techniques to forecast dengue fever outbreaks in Sudan. The purpose of this work is just to employ the fractional-order differential operator Caputo-Fabrizio (CF) and the fractional Euler approach to explain the dynamics of the dengue epidemic outbreak in Sudan. The second purpose of this work is to make comparisons of achievement of these algorithms and employ the most accurate methods to predict dengue fever outbreaks in Sudan.

This research is organized as follows: Section 1 introduces Dengue fever in Sudan; Section 2 gives crucial concepts required to understand

the remainder of the analysis; Section 3 Formulation of a Fractional Dengue Fever Model; Section 4 is devoted to building an epidemiological model for dengue disease using the Caputo-Fabrizio derivative, as well as considering the model's existence and uniqueness under the CF operator; section5 discusses numerical analysis and simulation; and Section 6 examines the performance of the models under discussion based on computing the errors of the methods, Section 7 result and discussion. And section 8 presents the key findings of the current research project.

2. Preliminaries

Definition 1 Riemann-Liouville fractional integral (RLI) operator of order α>0 for a function y(τ) is given by [16]:

$$D^{\alpha}y(t) \coloneqq \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}y^n(\tau)d\tau. = I^{n-\alpha}y^n(t), t>0 \quad (1)$$

Definition 2 The fractional integral (FI) of order α of a function f is defined as [17]

$$I_t^{\gamma}y(t) \coloneqq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}y(t) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t y(\tau)d\tau. t \geq 0, 0<\alpha<1 \quad (2)$$

Definition 3 Caputo derivative of order $0 \leq n-1 < \alpha < n$ with the lower limit zero for a function $y(\tau)$ is given by [18]:

$$I^{\alpha}y(t) \coloneqq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau.\ t>0 \quad (3)$$

Definition 4 For $y\epsilon H^1(0,t), t>0$, $T>0, \alpha\epsilon(0,1]$ Then the CF fractional operator [16] is given by

$${}^{CF}_{0}D_t^{\alpha}y(t) \coloneqq \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t y(\tau)exp\left(-\alpha\frac{t-\tau}{1-\alpha}\right)d\tau.\ 0<\alpha<1 \quad (4)$$

In this expression $B(\alpha)$ satisfies the condition $B(0)=B(1)=1$.

Definition 5 The Mittag-Leffler function (MLF) is a generalization of the exponential function. This function can be expressed as follows:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)} \quad (5)$$

Definition 6 For $y\epsilon H^1(0,t), t>0$, $T>0, \alpha\epsilon(0,1]$ Then the AB fractional operator [19] y(t) in the Riemann–Liouville is given by

$${}^{AB}_{0}D_t^{\alpha}y(t) \coloneqq \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t y(\tau)E_{\alpha}\left(\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau.\ 0<\alpha<1. \quad (6)$$

In this expression $B(\alpha)$ satisfies the condition $B(0)=B(1)=1$.

Definition 7 For y ∈ H1(0, T), T > 0Then the AB fractional operator [19] y(t) in the Caputo sense is given by

$${}^{AB}_{0}D_t^{\alpha}y(t) \coloneqq \frac{B(\alpha)}{1-\alpha}\int_0^t \frac{dy(\tau)}{d\tau}E_{\alpha}\left(\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau.\ 0<\alpha<1. \quad (7)$$

In this expression $B(\alpha)$ satisfies the condition $B(0)=B(1)=1$.

Definition 8: Let $0<\alpha<1$ and the fractional CF derivative is expressed as

$${}^{CF}_{0}D_t^{\alpha}y(t) = h(t) \quad (8)$$

3. Model Formulation:

A infectivity model of dengue fever in mathematical formulation is premised on a number of propositions, along with the following: The overall total number of persons (N_h) and mosquitoes (N_m) is constant, the rate of birth and morbidity are equal, births in mosquitos and humans for every category enter the suspected group, each person in the population is inclined to possess the same count of mosquito nibbles, and the infected mosquito is likely to sting each component of the data

provided. Table 1 shows the variables incorporated into the dengue fever illness framework. The above-mentioned model may be consumed in the form of a mathematical framework that is a host-vector interaction prototype, which is the following Fractional Differential approach:

Table 1 Definition of infectivity model components:

Variable	Description
$N_h(t)$	Overall number of people (constant ≈ 44 million)
$S_h(t)$	Susceptible humans
$I_h(t)$	Infected humans
$R_h(t)$	Recovery humans
$S_m(t)$	Susceptible female mosquitoes
$I_m(t)$	Infected female mosquitoes
Parameter	**Description**
μ_h	Human mortality rate for every person
μ_m	Corresponding value for the mosquitoes
γ_h	Recovery rate of the humans
B	The biting rate
β_{mh}	The likelihood that human to mosquito transfer may occur
β_{hm}	The likelihood that mosquito to human transfer may occur

$$\begin{cases} \frac{dS_h}{dt} = \mu_h N_h - \left(B\beta_{mh}\frac{I_h}{N_h} + \mu_h\right)S_h, \\ \frac{dI_h}{dt} = B\beta_{mh}\frac{S_h I_h}{N_h} - (\gamma_h + \mu_h)I_h, \\ \frac{dR_h}{dt} = \gamma_h I_h - \mu_h R_h, \\ \frac{dS_m}{dt} = \mu_m N_m - \left(B\beta_{mh}\frac{I_h}{N_h} + \mu_m\right)S_m, \\ \frac{dI_m}{dt} = B\beta_{mh}\frac{I_h}{N_h}S_m - \mu_m I_m \end{cases} \quad (9)$$

we expand the model (9) by employing the newly proposed CF derivatives with variable order $\alpha(t)$.

4. The CF derivative model:

In this part we apply the CF derivative to model (9). The fractional model is attained by replacing the classical derivative by the operator ${}^{CF}_{0}C D_t^{\alpha(t)}$:

4.1 Variable-order of fractional model:

$$\begin{aligned} {}^{CF}_{0}D_0^{\lambda} S_h &= \theta_1 - (\theta_2 + \theta_3)S_h, \\ {}^{CF}_{0}D_0^{\lambda} I_h &= \theta_2 S_h - (\theta_4 + \theta_3)I_h, \\ {}^{CF}_{0}D_0^{\lambda} R_h &= \theta_4 I_h - \theta_3 R_h, \\ {}^{CF}_{0}D_0^{\lambda} S_m &= \theta_6 - (\theta_2 + \theta_5)S_m, \\ {}^{CF}_{0}D_0^{\lambda} I_m &= \theta_2 S_m - \theta_5 I_m \end{aligned} \quad (10)$$

Where:

$$\theta_1 = \mu_h N_h,\ \theta_2 = B\beta_{mh}\frac{I_h}{N_h} = B\beta_{hm}\frac{I_h}{N_h},\ ,\ \theta_3 = \mu_h \quad , \quad \theta_4 = \gamma_h, \theta_5 = \mu_m \quad , \quad \theta_6 = \mu_m N_m \quad (11)$$

With conditions:

$$S_h(0) = c_1, I_h(0) = c_2, R_h(0) = c_3, S_m(0) = c_4, I_m(0) = c_5 \quad (12)$$

4.2 Existence and uniqueness of Solutions:

This section examines the existence and uniqueness of the fractional model solution with exponential law. In natural sciences, it is critical to understand the existence and uniqueness of any mathematical model's solution. As a result, we apply fixed point theory to investigate the existence and uniqueness of the solution of a fractional model [20-22]. We apply the FI operator (2) to Eq. (10),(11) and Eq. (12), which gives:

$$S_h(t) - S_h(0) = {}^{CF}_{0}I_t^{\gamma}(\theta_1 - (\theta_2 + \theta_3)S_h) \quad (13)$$

$$I_h(t) - I_h(0) = {}^{CF}_{0}I_t^{\gamma}(\theta_2 S_h - (\theta_4 + \theta_3)I_h) \quad (14)$$

$$R_h(t) - R_h(0) = {}^{CF}_{0}I_t^{\gamma}(\theta_4 I_h - \theta_3 R_h) \quad (15)$$

$$S_m(t) - S_m(0) = {}^{CF}_{0}I_t^{\gamma}(\theta_6 - (\theta_2 + \theta_5)S_m) \quad (16)$$

$$I_m(t) - I_m(0) = {}^{CF}_{0}I_t^{\gamma}(\theta_2 S_m - \theta_5 I_m) \quad (17)$$

Using the notation suggested in [23], we have

$$S_h(t) - S_h(0) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}(\theta_1 - (\theta_2 + \theta_3)S_h) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t(\theta_1 - (\theta_2 + \theta_3)S_h)d\tau \quad (18)$$

$$I_h(t) - I_h(0) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}(\theta_2 S_h - (\theta_4 + \theta_3)I_h) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t(\theta_2 S_h - (\theta_4 + \theta_3)I_h)d\tau \quad (19)$$

$$R_h(t) - R_h(0) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}(\theta_4 I_h - \theta_3 R_h) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t(\theta_4 I_h - \theta_3 R_h)d\tau \quad (20)$$

$$S_m(t) - S_m(0) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}(\theta_6 - (\theta_2 + \theta_5)S_m) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t(\theta_6 - (\theta_2 + \theta_5)S_m)d\tau \quad (21)$$

$$I_m(t) - I_m(0) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}(\theta_2 S_m - \theta_5 I_m) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t(\theta_2 S_m - \theta_5 I_m)\,d\tau \quad (22)$$

We express

$$f_1(t, S_h) = \theta_1 - (\theta_2 + \theta_3)S_h \quad (23)$$

$$f_2(t, I_h) = \theta_2 S_h - (\theta_4 + \theta_3)I_h \quad (24)$$

$$f_3(t, R_h) = \theta_4 I_h - \theta_3 R_h \quad (25)$$

$$f_4(t, S_m) = \theta_6 - (\theta_2 + \theta_5)S_m \quad (26)$$

$$f_5(t, I_m) = \theta_2 S_m - \theta_5 I_m \quad (27)$$

Theorem 1 the kernelsf_1,f_2,and f_3 satisfy the Lipschitz condition and contraction if

$$0 \le \theta_2 + \theta_3 < 1$$

Proof we initiate with f_1. For two functions x and x_1, we have

$$\|f_1(t, S_h) - f_1(t, S_{h_1})\| = \|(\theta_2 + \theta_3)(s_h(t) - s_{h1}(t))\|$$

$$\le (\theta_2 + \theta_3)\|(s_h(t) - s_{h1}(t))\|$$

$$\le b_1\|(s_h(t) - s_{h1}(t))\| \quad (28)$$

Taking $b_1 = \theta_2 + \theta_3$ where $\|S_h(t)\| \le a_1, \|I_h(t)\| \le a_2, \|S_m(t)\| \le a_3, \|I_m(t)\| \le a_3, \|s_h(t)\| \le a_3$ are bounded functions, we have

$$\|f_1(t, S_h) - f_1(t, S_{h_1})\| \le b_1\|s_h(t) - s_{h1}(t)\| \quad (29)$$

Thus the Lipschitz condition is satisfied forf_1. Furthermore, if
$0 \le \theta_2 + \theta_3 < 1$, then it is also a contraction. Similarly, we can prove that the kernels f_1 , f_2 , and f_3 satisfy the Lipschitz conditions

$$\|f_1(t, I_h) - f_1(t, I_{h_1})\| \le b_2\|I_h(t) - I_{h1}(t)\| \quad (30)$$

$$\|f_1(t, R_h) - f_1(t, R_{h_1})\| \le b_3\|R_h(t) - R_{h1}(t)\| \quad (31)$$

$$\|f_1(t, S_m) - f_1(t, S_{m1})\| \le b_4\|S_m(t) - S_{m1}(t)\| \quad (32)$$

$$\left\|f_1(t,I_m)-f_1\left(t,I_{m_1}\right)\right\| \leq b_5\|I_m(t)-I_{m1}(t)\| \quad (33)$$

using the notations of the earlier stated kernels, (18-22) reduces to the system

$$S_h(t)-S_h(0)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1(t,S_h)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1(\tau,S_h)\right)d\tau \quad (34)$$

$$R_h(t)-R_h(0)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1\left(t,R_h\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1(\tau,R_h)\right)d\tau \quad (35)$$

$$I_h(t)-I_h(0)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1(t,I_h)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1(\tau,I_h)\right)d\tau \quad (36)$$

$$S_m(t)-S_h(0)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1(\tau,S_m)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1(\tau,S_m)\right)d\tau \quad (37)$$

$$I_m(t)-I_m(0)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1(t,I_m)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1(\tau,I_m)\right)d\tau \quad (38)$$

Next, we construct the following recursive formulas:

$$S_{hn}(t)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1\left(t,s_{h(n-1)}\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1\left(t,I_{h(n-1)}\right)\right)d\tau \quad (39)$$

$$R_{hn}(t)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1\left(t,R_{h(n-1)}\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1\left(t,R_{h(n-1)}\right)\right)d\tau \quad (40)$$

$$I_h(t)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1\left(t,I_{h(n-1)}\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1\left(t,I_{h(n-1)}\right)\right)d\tau \quad (41)$$

$$S_{mn}(t)-=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1\left(t,S_{m(n-1)}\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1\left(t,S_{m(n-1)}\right)\right)d\tau \quad (42)$$

$$I_{mn}(t)-=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f_1\left(t,I_{m(n-1)}\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1\left(t,I_{m(n-1)}\right)\right)d\tau \quad (43)$$

With the initial conditions:

$$S_{h_0}=S_h(0),I_{h_0}=I_h(0),R_0=R_h(0),S_{m_0}=S_m(0),I_{m_0}=I_m(0) \quad (44)$$

We express the difference between the succession terms as

$$g_{1n}(t)=S_{hn}(t)-S_{h(n-1)}(t)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1\left(t,s_{h(n-1)}\right)-f_1\left(t,s_{h(n-2)}\right)\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1\left(t,s_{h(n-1)}\right)-f_1\left(t,s_{h(n-2)}\right)\right)d\tau \quad (45)$$

$$g_{2n}(t)=I_{hn}(t)-I_{h(n-1)}(t)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1\left(t,I_{h(n-1)}\right)-f_1\left(t,I_{h(n-2)}\right)\right)+\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1\left(t,I_{h(n-1)}\right)-\right.$$

$f_1\left(t, I_{h(n-2)}\right)\Big)\, d\tau$

(45)

$$g_{3n}(t) = R_{hn}(t) - R_{h(n-1)}(t) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1\left(t, R_{h(n-1)}\right) - f_1\left(t, R_{h(n-2)}\right)\right) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left(f_1\left(t, R_{h(n-1)}\right) - f_1\left(t, R_{h(n-2)}\right)\right) d\tau$$

(46)

$$g_{4n}(t) = S_{mn}(t) - S_{m(n-1)}(t) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1\left(t, S_{m(n-1)}\right) - f_1\left(t, S_{m(n-2)}\right)\right) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left(f_1\left(t, S_{m(n-1)}\right) - f_1\left(t, S_{m(n-2)}\right)\right) d\tau$$

(46)

$$g_{5n}(t) = I_{mn}(t) - I_{m(n-1)}(t) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1\left(t, I_{m(n-1)}\right) - f_1\left(t, I_{m(n-2)}\right)\right) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left(f_1\left(t, I_{m(n-1)}\right) - f_1\left(t, I_{m(n-2)}\right)\right) d\tau$$

(47)

It is worth observing that

$$S_{hn}(t) = \sum_{i=0}^{n} g_{1i}(t)\,, I_{hn}(t) = \sum_{i=0}^{n} g_{2i}(t)\,, R_{hn}(t) = \sum_{i=0}^{n} g_{3i}(t)\,, S_{mn}(t) = \sum_{i=0}^{n} g_{1i}(t)\,, I_{mn}(t) = \sum_{i=0}^{n} g_{2i}(t)\,,$$

(48)

Now we easily obtain the following result:

$$\|g_{1n}(t)\| = \left\|s_{hn}(t) - S_{h(n-1)}(t)\right\| = \left\|\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1\left(t, s_{h(n-1)}\right) - f_1\left(t, s_{h(n-2)}\right)\right) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left(f_1\left(t, s_{h(n-1)}\right) - f_1\left(t, s_{h(n-2)}\right)\right) d\tau\right\|$$

(49)

Applying the triangle inequality to Eq. (49), we get

$$\left\|s_{hn}(t) - S_{h(n-1)}(t)\right\| \le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left\|f_1\left(t, s_{h(n-1)}\right) - f_1\left(t, s_{h(n-2)}\right)\right\| + \left\|\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left(f_1\left(t, s_{h(n-1)}\right) - f_1\left(t, s_{h(n-2)}\right)\right) d\tau\right\|$$

(49)

It is already proved that the kernels satisfy the Lipschitz condition, so Eq. (49) gives

$$\left\|s_{hn}(t) - S_{h(n-1)}(t)\right\| \le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left\|s_{hn} - s_{h(n-1)}\right\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left\|s_{hn} - s_{h(n-1)}\right\| d\tau$$

(50)

we get ,

$$\|g_{1n}(t)\| \le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_1 \|g_{1n}(t)\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_1 \int_0^t \left\|g_{1(n-1)}(\tau)\right\| d\tau \qquad (51)$$

Using the same process, we derive the following results

$$\|g_{2n}(t)\| \leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_2 \|g_{2n}(t)\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_2 \int_0^t \left\|g_{2(n-1)}(\tau)\right\| d\tau \quad (52)$$

$$\|g_{3n}(t)\| \leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_3 \|g_{3n}(t)\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_3 \int_0^t \left\|g_{3(n-1)}(\tau)\right\| d\tau \quad (53)$$

$$\|g_{4n}(t)\| \leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_4 \|g_{4n}(t)\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_4 \int_0^t \left\|g_{4(n-1)}(\tau)\right\| d\tau \quad (54)$$

$$\|g_{5n}(t)\| \leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_5 \|g_{4n}(t)\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_5 \int_0^t \left\|g_{4(n-1)}(\tau)\right\| d\tau \quad (55)$$

Taking (51-55) into account, we obtain the existence of the solution of the considered model.

Theorem 2 The Fractional Dengue Fever model involving the CF fractional operator expressed in Eqs. (10) (11) and Eq. (12), has a solution if there exists t_0 such that

$$\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_1 + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_1 t_0 < 1 \quad (56)$$

Proof As we know, the functions $x(t), y(t), and\ z(t)$ are bounded. Using the results presented in Eqs. (51-55) and utilizing the recursive algorithm, we get

$$\|g_{1n}(t)\| \leq \|S_{hn}(t)\| \left[\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_1 + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_1 t_0\right]^n \quad (57)$$

$$\|g_{2n}(t)\| \leq \|I_{hn}(t)\| \left[\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_2 + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_2 t_0\right]^n \quad (58)$$

$$\|g_{3n}(t)\| \leq \|R_{hn}(t)\| \left[\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_3 + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_3 t_0\right]^n \quad (59)$$

$$\|g_{4n}(t)\| \leq \|S_{mn}(t)\| \left[\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_4 + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_4 t_0\right]^n \quad (60)$$

$$\|g_{5n}(t)\| \leq \|I_{mn}(t)\| \left[\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_5 + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} b_5 t_0\right]^n \quad (61)$$

Hence the solution of the considered model exists and is continuous. Now, to show that Eq. (50) is a solution of the model Eqs. (10) (11) and (12) we take

$$S_h(t) - S_h(0) = S_{hn}(t) - A_n(t) \quad (62)$$

$$I_h(t) - x(0) = I_{hn}(t) - B_n(t) \quad (63)$$

$$R_h(t) - I_h(0) = R_{hn}(t) - C_n(t) \quad (64)$$

$$S_m(t) - S_m(0) = S_{mn}(t) - D_n(t) \quad (65)$$

$$I_m(t) - I_m(0) = I_{mn}(t) - E_n(t) \quad (66)$$

Thus we have

$$\|A_n(t)\| = \left\|\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1(t,S_h) - f_1\left(t,S_{h(n-1)}\right)\right) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left(f_1(\tau,S_h) - f_1\left(\tau,S_{h(n-1)}\right)\right)d\tau\right\|$$

$$\leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left\|f_1(t,S_h) - f_1\left(t,S_{h(n-1)}\right)\right\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t \left\|f_1(t,S_h) - f_1\left(t,S_{h(n-1)}\right)\right\|d\tau$$

$$\leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}b_1\left\|S_h - S_{h(n-1)}\right\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}b_1\left\|S_h - S_{h(n-1)}\right\|t \quad (67)$$

Using this process recursively [], we get

$$\|A_n(t)\| \leq \left(\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}t\right)^{n+1} {b_1}^{n+1}a_1 \quad (68)$$

Then at t_0, we have

$$\|A_n(t)\| \leq \left(\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}t_0\right)^{n+1} {b_1}^{n+1}a_1 \quad (69)$$

Taking the limit on Eq. (69) as n tends to infinity gives

$\|A_n(t)\| \to 0$ Similarly, we get $\|B_n(t)\| \to 0$, $\|C_n(t)\| \to 0$

The existence theorem's proof is now complete.

Next, we prove the uniqueness of a solution of the fractional model Eqs. (10) (11) and (12).

Let's suppose that there is another system of solutions of the fractional model the fractional model $S_h{}^*(t), I_h{}^*(t), R_h{}^*(t), S_m{}^*(t) and\ I_m{}^*(t)$ Then

$$S_h(t) - S_h{}^*(t) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(f_1(t,S_h) - f_1(t,S_h{}^*)\right) + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\left(f_1(\tau,S_h) - f_1(\tau,S_h{}^*)\right)d\tau \quad (70)$$

Taking the norms gives

$$\|S_h(t) - S_h{}^*(t)\| \leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\|f_1(t,S_h) - f_1(t,S_h{}^*)\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\|f_1(t,S_h) - f_1(t,S_h{}^*)\|d\tau \quad (71)$$

Employing the results presented in (29-33), (4.6) we get

$$\|S_h(t) - S_h{}^*(t)\| \leq \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\|S_h(t) - S_h{}^*(t)\| + \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\int_0^t\|S_h(t) - S_h{}^*(t)\|d\tau \quad (72)$$

which gives

$$\|S_h(t) - S_h{}^*(t)\|\left(1 - \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}b_1 - \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}b_1 t\right) \leq 0 \quad (73)$$

Theorem 3 The Fractional Dengue Fever model Eqs. (10) (11) and (12)

has a unique solution if

$$1 - \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\gamma_1 - \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\gamma_1 t > 0 \quad (74)$$

Proof from Eq. (73) and properties of a norm in Eq. (74) gives:

$$\|S_h(t) - S_h^*(t)\| = 0 \quad (75)$$

Thus we can see that $S_h(t) = S_h^*(t)$.

Using a similar procedure, we easily prove that

$$S_h(t) = S_h^*(t), I_h(t) = I_h^*(t), R_h(t) = R_h^*(t), S_m(t) = S_m^*(t), I_m(t) = I_m^*(t) \quad (76)$$

The Fractional Dengue Fever model Eqs. (10) (11) and (12) has a unique solution.

5. The Basic Reproduction Number:

In this part we expected value of the secondary infections rate per time unit is denoted by R_0, and the basic reproduction number is a baseline metric in epidemiology. Based on the Fractional Model of equation (10), We have two infected classes $I_h(t), I_m(t)$.

$${}_0^{CF}D_0^{\lambda} I_h = \theta_2 S_h - (\theta_4 + \theta_3) I_h \quad (77a)$$

$${}_0^{CF}D_0^{\lambda} I_m = \theta_2 S_m - \theta_5 I_m \quad (77b)$$

We assume that the Hazard rate of infection is a constant $\beta(t)$ taking the maximum value $\beta(0) = \beta_0$ and the minimum value $\beta(t^*) = \beta^*$.

Let $x = (I_h, I_m)$ .and rewrite the system of equation (77) for the susceptible and infected classes in the general from

$$\frac{dx}{dt} = f(x) - v(x) \quad (78)$$

Where

$$f(x) = \begin{bmatrix} B\beta_{mh} & \frac{S_h I_h}{N_h} \\ B\beta_{mh} & \frac{I_h S_m}{N_h} \end{bmatrix}, v(x) = \begin{bmatrix} \beta_{mh}\frac{S_h}{N_h} & 0 \\ \beta_{mh}\frac{S_m}{N_h} & 0 \end{bmatrix} \quad (79)$$

Now the Jacobian of $f(x)$ and $v(x)$ of the disease free equilibrium point is

$$F = \begin{bmatrix} B\beta_{mh}\frac{S_h}{N_h} & 0 \\ B\beta_{mh}\frac{S_m}{N_h} & 0 \end{bmatrix} \quad (80)$$

$$V = \begin{bmatrix} (\gamma_h + \mu_h) I_h & 0 \\ 0 & \mu_m \end{bmatrix} \quad (81)$$

Therefore

$$V^{-1} = \begin{bmatrix} \frac{1}{(\gamma_h + \mu_h) I_h} & 0 \\ 0 & \frac{1}{\mu_m} \end{bmatrix} \quad (82)$$

By use Eq. (11) we have

$$R_0 = \rho(FV^{-1}) = \frac{1}{\mu_m} \quad (83)$$

6. Stability analysis of the model:

Theorem 3 The Fractional Model of equation (10) is locally stable related to the free equilibrium point E_1 if R_0>1, and unstable if R_0<1.

Proof: The Jacobian matrix [24] with respect to the Fractional Model of equation (10) is given by:

$$\boldsymbol{J} = \begin{bmatrix} -B\beta_{mh}\frac{I_h}{N_h} - \mu_h & -B\beta_{mh}\frac{S_h}{N_h} & 0 & 0 & 0 \\ B\beta_{mh}\frac{I_m}{N_h} & -(\gamma_h + \mu_h) & 0 & 0 & 0 \\ 0 & \gamma_h & -\mu_h & 0 & 0 \\ 0 & -B\beta_{mh}\frac{S_m}{N_h} & 0 & -\mu_m - B\beta_{hm}\frac{I_h}{N_h} & 0 \\ 0 & B\beta_{mh}\frac{S_m}{N_h} & 0 & B\beta_{hm}\frac{I_h}{N_h} & -\mu_m \end{bmatrix} \quad (84)$$

At, E_1 the Jacobin matrix becomes

$$\boldsymbol{J}(E_1) = \begin{bmatrix} -\mu_h & -B\beta_{mh} & 0 & 0 & 0 \\ 0 & -(\gamma_h + \mu_h) & 0 & 0 & B\beta_{mh} \\ 0 & B\beta_{mh}\gamma_h\frac{S_h}{N_h} & -\mu_h & 0 & 0 \\ 0 & -B\beta_{mh} & 0 & -\mu_m & 0 \\ 0 & B\beta_{hm} & 0 & 0 & -\mu_m \end{bmatrix} \quad (85)$$

Thus, we get the eigenvalues are

$$\sigma_1 = -\mu_h, \sigma_2 = -\mu_h, \sigma_3 = -\mu_m, \sigma_4 = \frac{-(\gamma_h+\mu_h+\mu_m)}{2} - k, \sigma_5 = k - \frac{(\gamma_h+\mu_h+\mu_m)}{2} \quad (86)$$

Where

$$k = \frac{1}{2}\sqrt{(4\mu_m\beta_{mh}\beta_{hm}B^2)/N_h + {\gamma_h}^2 + 2\gamma_h\mu_h - 2\gamma_h\mu_m + {\mu_h}^2 - 2\mu_h\mu_m + {\mu_m}^2} \quad (87)$$

If $2k < \gamma_h + \mu_h + \mu_m$, $\sigma_5 < 0$ and $R_0 < 1$, then the Fractional Model is Stable for, $R_0 < 1$, Hence the Fractional Model of equation (10) is unstable for $R_0 > 1$.

7. Numerical scheme and simulations:

In this section, simulation of the Fractional infectivity model of dengue fever in Sudan obtained using MATLAB programs to show the dynamics of an epidemic and to study different strategies. In general, two approaches to controlling the pandemic can be considered. The first based on the number of mosquitos, while the second based on the total of susceptible people. the initial values of the model are:

$$S_h(0) = 3326, I_h(0) = 482, R_h(0) = 0, S_m(0) = 117600, I_m(0) = 5600$$

used in the simulation are based on the information on dengue fever in Sudan that is displayed in Table 2. The basic reproduction number R_0 is obtained from equation (83).

Table 2 Rest of epidemiological parameters of the proposed model

Parameter	Range of values	References
N_h	44909351	Constant
N_m	168000	Estimated
μ_h	0.0000031	Estimated
μ_m	0.1	[16]
γ_h	2-10 days – [1/3]	[16]
B	0.7	[16]
β_{mh}	0.36	[16]
β_{hm}	0.36	[16]

The approximate solution of the fractional model obtains by using two-step fractional Adam-Bashforth approach for the CF derivatives.

Figure 1 Simulations for (Susceptible, Infected, Recovered) for human beings.

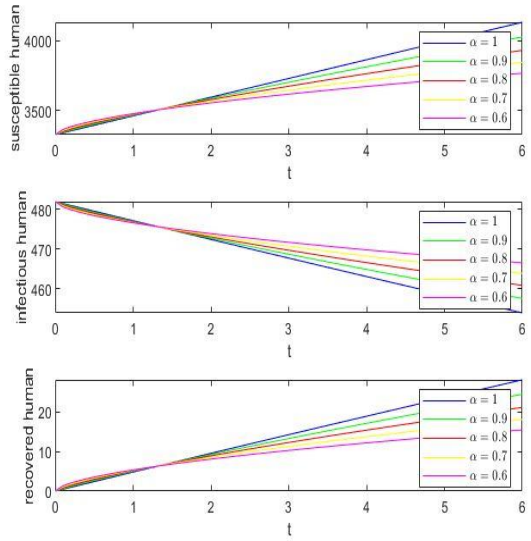


Figure 2 Simulations of dengue fever for Susceptible mosquitoes, Infected mosquitoes.

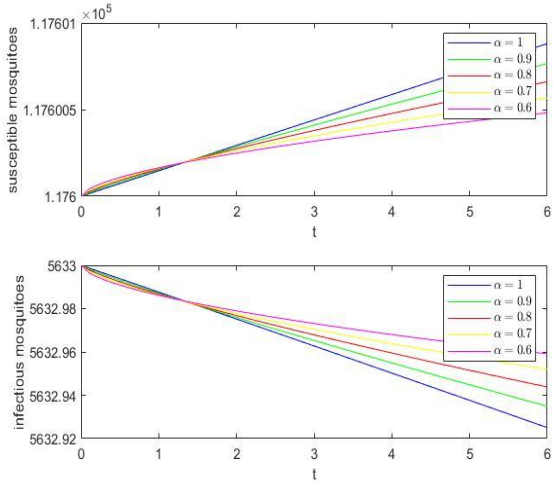
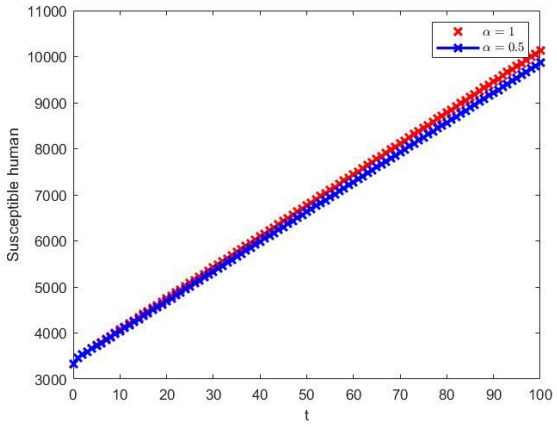
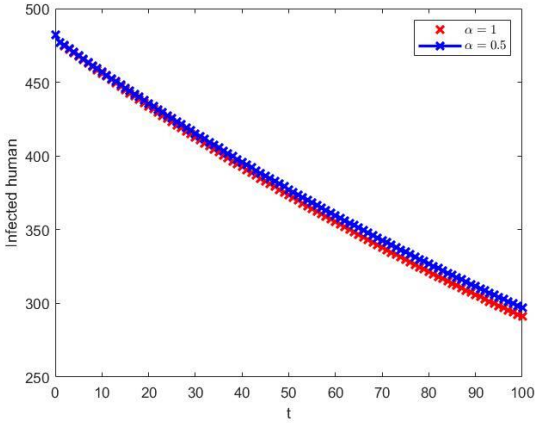


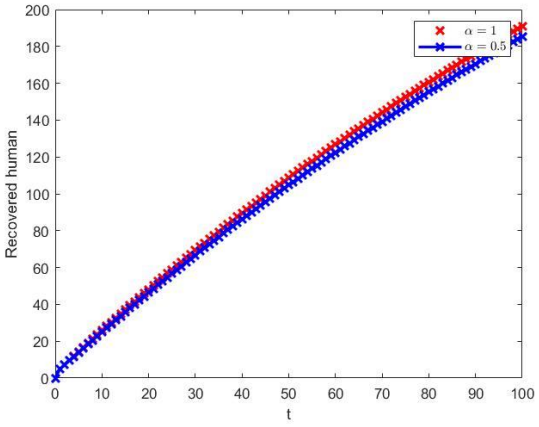
Fig. 3. compare between CF sense and α=0.5.



a



b



c

8. Result and discussion

The Eigenvalues determine the stability analysis of the obtained equilibrium values. Eigenvalues of the infectivity model of dengue disease are determined based on equations (36). Based on the prevalence of dengue fever in Sudan, the basic reproduction number 〖 R〗 _0=10>1,indicates that one sick person can spread the disease to up to 10 more people. In other words, dengue fever is endemic among Sudanese people. The dynamics of dengue fever for (Susceptible, Infected, Recovered) for human beings and (Susceptible, Infected) for mosquitoes are depicted in Figs. 1 and Figs. 2. The Fig. 3. display the compare between CF sense and α=0.5.

9. Conclusion

In this study, we used the Adam-Bashforth technique and CF derivatives to get an approximation solution to the fractional order dengue fever model.

For the simulations of classical and fractional-order models with the CF operator, real statistical data regarding the dengue fever outbreak in Sudan (2022) was used. At fractional order α=0.5, we compared the classical and CF senses. The fundamental reproduction number, which is based on the prevalence of dengue fever in Sudan, implies that one infected person can spread the disease to up to 10 additional persons. To put it another way, dengue fever is extremely common among Sudanese people.

Reference

[1] Mohammed A. Soghaier, Syed F. Mahmood al, Factors associated with dengue fever IgG zero-prevalence in South Kordofan State, Sudan, in 2012: Reporting prevalence ratios, Journal of Infection and Public Health, Volume 7, Issue 1,2014, Pages 54-61, https://doi.org/10.1016/j.jiph.2013.07.008.

[2] Malik A, Earhart K, Mohareb E, et al. Dengue hemorrhagic fever outbreak in children in Port Sudan. J Infect Public Health. 2011;4(1):1-6. doi:10.1016/j.jiph.2010.08.001

[3] Ahmed A, Dietrich I, LaBeaud AD, Lindsay SW, Musa A, Weaver SC. Risks and Challenges of Arboviral Diseases in Sudan: The Urgent Need for Actions. Viruses. 2020;12(1):81. Published 2020 Jan 9. doi:10.3390/v12010081

[4] Armstrong, C. Dengue Fever. Public Health Rep. (1896–1970) 1923, 38, 1750–1784.

[5] Balfour, A.; Archibald, R.G. Wellcome Research Laboratories (Khartoum), London School of Hygiene and Tropical Medicine. In Review of Some of the Recent Advances in Tropical Medicine, Hygiene and Tropical Veterinary Science; Forgotten Books: London, UK, 1908; p. 262.

[6] Balfour, A. Notes on the differential leucocyte count, with special reference to dengue fever. Ibid 1907, 10, 113.

[7] Humphrey JM, Cleton NB, Reusken CB, Glesby MJ, Koopmans MP, Abu-Raddad LJ. Dengue in the Middle East and North Africa: A Systematic Review. PLoS Negl Trop Dis. 2016;10(12):e0005194. Published 2016 Dec 7. doi:10.1371/journal.pntd.0005194

[8] Hamid Z, Hamid T, Alsedig K, et al. Molecular Investigation of Dengue Virus Serotype 2 Circulation in Kassala State, Sudan. Jpn J Infect Dis. 2019;72(1):58-61. doi:10.7883/yoken.JJID.2018.267

[9] Hyams KC, Oldfield EC, Scott RM, et al. Evaluation of febrile patients in Port Sudan, Sudan: isolation of dengue virus.

Am J Trop Med Hyg. 1986;35(4):860-865. doi:10.4269/ajtmh.1986.35.860

[10] Abdalla TM, Karsany MS, Ali AA. Correlation of measles and dengue infection in Kassala, Eastern Sudan. J Med Virol. 2015;87(1):76-78. doi:10.1002/jmv.24001

[11] Seidahmed OM, Siam HA, Soghaier MA, et al. Dengue vector control and surveillance during a major outbreak in a coastal Red Sea area in Sudan. East Mediterr Health J. 2012;18(12):1217-1224.

[12] Malik A, Earhart K, Mohareb E, et al. Dengue hemorrhagic fever outbreak in children in Port Sudan. J Infect Public Health. 2011;4(1):1-6. doi:10.1016/j.jiph.2010.08.001

[13] Ahmed A, Ali Y, Elmagboul B, et al. Dengue Fever in the Darfur Area, Western Sudan. Emerg Infect Dis. 2019;25(11):2126. doi:10.3201/eid2511.181766

[14] Ahmed A, Elduma A, Magboul B, Higazi T, Ali Y. The First Outbreak of Dengue Fever in Greater Darfur, Western Sudan. Trop Med Infect Dis. 2019;4(1):43. Published 2019 Mar 1. doi:10.3390/tropicalmed4010043

[15] P.Pongsumpun, "Transmission model for dengue disease with and without the effect of extrinsic incubation period," KMITL Sci. Tech., vol. 6, pp. 74–82, 2006.

[16] I Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, in: Mathematics in Science and Engineering, vol. 198, 1999.

[17] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 1 (2) (2015) 73–85.

[18] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, Therm. Sci. 20 (2) (2016) 763–769.

[19] Losada, J.J. Nieto, Properties of a new fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 1 (2) (2015) 87–92.

[20] Atangana, A., Alkahtani, B.T.: Analysis of non-homogenous heat model with new trend of derivative with fractional order. Chaos Solitons Fractals 89, 566–571 (2016).

[21] Atangana, A., Alqahtani, B.S.T.: Analysis of the Keller–Segel model with a fractional derivative without singular kernel. Entropy 17, 4439–4453 (2015).

[22] Kumar, D., Singh, J., Baleanu, D.: Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel. Physica A 492, 155–167 (2018).

[23] Losada, J.J., Nieto, J.: Properties of the new fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1, 87–92 (2015).

[24] P. Van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002), 29-48.